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Asset Prices in a Production Economy with Long Run and Idiosyncratic Risk*

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Abstract

This paper studies risk premia in an incomplete-markets economy with households facing idiosyncratic consumption risk. If the dispersion of idiosyncratic risk varies over the business cycle and households have preference for early resolution of uncertainty, asset prices will be affected not only by news about current and expected future aggregate consumption (as in models with a representative agent), but also by news about current and future changes in cross-sectional distribution of individual consumption. I investigate whether this additional effect can help to explain high risk premia in a production economy, where the aggregate consumption process is endogenous and thus can potentially be affected by the presence of idiosyncratic risk. Analyzing a neoclassical growth model combined with Epstein-Zin preferences and a tractable form of household heterogeneity, I find that countercyclical idiosyncratic risk increases the risk premium, but also effectively lowers willingness of households for intertemporal substitution and thus changes dynamics of aggregate consumption. Nevertheless, with the added flexibility of Epstein-Zin preferences, it is possible to both increase risk premia and maintain the same dynamics of quantities if we allow for higher intertemporal elasticity of substitution at the individual level.

JEL: E13, E21, E44, G12

Keywords: incomplete markets, idiosyncratic risk, production economy, risk premium

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1 Introduction

Explaining joint dynamics of both macroeconomic quantities and asset prices within the context of a microfounded general equilibrium model remains an active area of economic research. This paper contributes to that effort by constructing a tractable model of a production economy that combines recursive utility with preference for early resolution of uncertainty and time-varying uninsurable idiosyncratic risk, and investigates its macroeconomic and asset pricing properties.

Individually, each of these elements have been studied previously as a possible solution to the well-known failures of a standard representative-agent model with power utility in explaining observed equity premium and interest rate[1]. When households have recursive preferences (Kreps and Porteus 1978; Epstein and Zin 1989), which break the link between risk aversion and elasticity of intertemporal substitution and allow for preference for early resolution of uncertainty, their marginal utility depends not only on current consumption, but also on the continuation value which encodes expectations about future consumption. News regarding the level or volatility of future consumption thus becomes an additional priced factor, as in the long-run risk model of Bansal and Yaron (2004) and in the production economy[2] of Kaltenbrunner and Lochstoer (2010). Another line of research has shown that when agents face incomplete markets and uninsurable shocks, the amount of risk they face can also affect asset prices if it changes over time, as in Constantinides and Duffie (1996) and Krusell and Smith (1997)[3].

Therefore, if agents have preference for early resolution of uncertainty and at the same time face idiosyncratic risk and incomplete markets, it follows that both current change in the amount of idiosyncratic risk, and also news about future such changes

[1] See e.g. Mehra and Prescott (1985), Weil (1989) and Hansen and Singleton (1982). A review of the literature is provided in e.g. Cochrane (2008) and Ludvigson (2013).

[2] Regarding asset pricing in production/DSGE models, see, e.g., survey by Kogan and Papanikolaou (2012). Among papers that study asset prices in production economies with recursive preferences are Tallarini (2000), Kaltenbrunner and Lochstoer (2010), Croce (2014), Rudebusch and Swanson (2012), van Binsbergen et al. (2012) and Campanale, Castro, and Clementi (2010).

[3] See also Mankiw (1986), Telmer (1993), Heaton and Lucas (1996), Krebs and Wilson (2004), Storesletten, Telmer, and Yaron (2007) and Pijoan-Mas (2007). Gomes and Michaelides (2008) also study a model with heterogeneity, production and recursive preferences, but their focus is primarily on the effects of limited participation and they do not model variation in either individual or aggregate risk over time. Empirical evidence is analyzed, e.g., by Cogley (2002, Brav, Constantinides, and Geczy (2002) and Balduzzi and Yao (2007), with somewhat mixed results.

enter the continuation value and thus affect asset prices. This presents the potential for interaction between the two mechanisms, studied in the context of an endowment economy in recent work by Constantinides and Ghosh (2017), Herskovic et al. (2016) and Schmidt (2014). However, matching asset prices in a production economy is harder than in endowment economies due to endogenous consumption process and the need to simultaneously match properties of quantities and prices. The main focus of this paper is therefore to look more closely at the interaction between the effects of varying idiosyncratic risk on macroeconomic dynamics and asset prices.

To illustrate the mechanism, I first construct a simple AK model with households having access to linear production technologies subject to heterogeneous rates of return on capital with time-varying variance. Assuming unit intertemporal elasticity of substitution, the model can be solved analytically and asset returns can be characterized by their exposure to news about current and future aggregate consumption and variance of idiosyncratic risk. A quantitative illustration suggests that omitting the last term could nontrivially underestimate the importance of overall long run risk for determining risk premia.

Next, I contruct a tractable model that embeds the Constantinides-Duffie framework within an otherwise standard real business cycle (RBC) model[4]. Individual household consumption growth is determined, in a reduced-form way, by aggregate consumption growth and idiosyncratic shock. With homothetic preferences and random walk in individual consumption, the model has a no-trade equilibrium in which each household consumes its income. The aggregate stochastic discount factor is determined by the cross-sectional average of individual intertemporal marginal rates of substitution, and is used by a representative firm to make choices about investment and dividends, which in turn determines aggregate consumption growth. Distribution of idiosyncratic shocks varies over time, possibly allowing for countercyclical variance (Storesletten, Telmer, and Yaron 2004) or procyclical skewness (Guvenen, Ozkan, and Song 2014).

The fact that there is no trade between households is somewhat unappealing (and thus resulting allocations should perhaps be interpreted rather as post-trade outcomes after households have smoothed out transitory shocks), yet it allows us to solve the

[4]A similar approach is used to analyze monetary policy in New-Keynesian models in recent papers by Braun and Nakajima (2012), Werning (2015) and Takahashi et al. (2016). In these setups, variation in idiosyncratic risk manifests itself in a similar way as discount rate shocks after aggregation. In a related study, Albuquerque et al. (2016) study the role of discount rate shocks in asset pricing.

model without keeping track of the distribution over individual savings, and thus avoid the need for numerically intensive computation. The model can be solved by standard perturbation methods and its linearized dynamics can be characterized semi-analytically. I find that the countercyclical idiosyncratic risk can raise risk premia, but also affects aggregate dynamics through its impact on saving and intertemporal smoothing incentives of households. The introduction of idiosyncratic risk leads to lower "effective" intertemporal elasticity of substitution on the aggregate level, resulting in more volatile and less predictable aggregate consumption growth. Inspecting the linearized solution suggests that the strength of this feedback depends on the cyclicality of idiosyncratic risk and household risk aversion.

On the other hand, thanks to the flexibility of Epstein-Zin preferences, it is, in principle, possible to recalibrate the discount rate and intertemporal elasticity of substitution (IES) parameters (to make households more willing to substitute consumption over time) in a way that compensates for the effect described above while risk premia remain higher. After suitable recalibration of the model, I find that introducing heterogeneity raises the price of risk (Sharpe ratio) by about a third. Decomposing the price of risk by its source (aggregate consumption or dispersion of individual shocks) and channel (short-run or long-run risk) shows that the long run idiosyncratic dispersion accounts for about 30 percent of the overall long run channel, which int turn accounts for more than half of the overall Sharpe ratio. The results are quite similar regardless of whether the variation in individual risk unfolds through cyclical variance or skewness.

The paper is organized as follows: section 2 presents a simple example to motivate introduction of recursive preferences, section 3 describes the model, while section 4 discusses calibration and results and section 5 concludes.

2 Simple Model

Analytic results which illustrate the interplay between idiosyncratic and long run risk can be obtained relatively easily in a simplified AK-like model where output is produced using a linear technology with capital as the only input. Previous literature using AK models to analyze asset prices in the presence of idiosyncratic risk includes Krebs and Wilson (2004), who focused on the case of log utility, and Toda (2014), who provides theoretical analysis for a class of similar models.

Time t is discrete and there is a continuum of agents indexed by i. Each agent enters the period with some stock of capital $K_{i,t}$ which is used for production according to $Y_{i,t} = A_{i,t}K_{i,t}$, subject to exogenous productivity process $A_{i,t}$ (which will have an idiosyncratic component and is thus indexed by i). Agents can also trade in risk-free one-period bonds, although the overall net supply of bonds is zero. Income obtained from production and bond holdings $B_{i,t}$ can be used for consumption $C_{i,t}$, stored as capital for the next period (for simplicity we shall assume full depreciation) or spent on new bonds. The budget constraint thus reads

$$C_{i,t} + K_{i,t+1} + P_t^b B_{i,t+1} = A_{i,t}K_{i,t} + B_{i,t},$$

where P_t^b is the bond price.

Agents have identical Epstein-Zin preferences with unit intertemporal elasticity of subtitution, so that their value function satisfies

$$V_{i,t} = C_{i,t}^{1-\beta} \left(E_t[V_{i,t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}} \right)^{\beta}.$$

Here parameter β controls time preference and γ is the coefficient of relative risk aversion. In the following, we shall focus on the empirically relevant case $\gamma > 1$, so that agents have preference for early resolution of uncertainty. Given the process for productivity, bond price and initial capital, each household will make its consumption-savings and portfolio choice to maximize the value function defined above.

We shall assume that the productivity has aggregate and idiosyncratic component:

$$\log(A_{i,t}) = \log(A_t) + \sqrt{x_t}\eta_{i,t} - \frac{x_t}{2}, \;\; \eta_{i,t} \sim N(0,1)$$

where idiosyncratic shocks $\eta_{i,t}$ are independent both across time and across households. Another exogenous process x_t denotes the cross-sectional variance of log productivity, which will fluctuate over time, and the last term ensures that the normalization $A_t = \tilde{E}[A_{i,t}]$ holds ($\tilde{E}[\,]$ will denote cross-sectional averages, conditional on realizations of aggregate variables).

The equilibrium of this economy turns out to be particularly simple:

- Since preferences are homothetic and the value function is linear in wealth, there is a separation between the consumption-saving decision and portfolio choices. Since idiosyncratic shocks are uncorrelated over time, the only source of heterogeneity is in differing levels of wealth, so that all households make the same portfolio choice. Given the zero net supply of bonds, the equilibrium must thus involve no trade in them, so that $\forall i, \forall t : B_{i,t} = 0$.

- Without bonds, all wealth comes from current production. With unit IES, the consumption choice will be a constant linear function of wealth, so that $C_{i,t} = \kappa Y_{i,t}$ and $K_{i,t} = (1-\kappa)Y_{i,t}$, where $\kappa = 1-\beta$.

Defining aggregates straightforwardly as cross-sectional averages (e.g. $K_t = \tilde{E}[K_{i,t}]$, etc.), aggregate dynamics can be summarized easily:

$$\begin{aligned} Y_t &= A_t K_t, \\ C_t &= \kappa Y_t, \\ K_{t+1} &= (1-\kappa)Y_t. \end{aligned}$$

Note that aggregate dynamics of quantities depends only on the aggregate productivity process A_t, not on the cross-sectional variance process x_t. If we denote logs in lowercase, we can also derive aggregate and individual consumption growth as

$$\begin{aligned} \Delta c_t &= \log(C_t/C_{t-1}) = \log((1-\kappa)A_t) = \log(1-\kappa) + a_t, \\ \Delta c_{i,t} &= \log(C_{i,t}/C_{i,t-1}) = \log((1-\kappa)A_{i,t}) = \log(1-\kappa) + a_t + \sqrt{x_t}\eta_{i,t} - \frac{x_t}{2}. \end{aligned}$$

The process for individual consumption thus has a similar form as in Constantinides and Duffie (1996).

Moving on to asset prices, although strictly speaking there is no aggregate capital, we can naturally define aggregate return to capital as an average payoff at time $t+1$ to one unit of good invested at time t, so that $R^k_{t+1} = A_{t+1}$. Return on bonds is then defined as $R^b_{t+1} = \frac{1}{P^b_t}$, and the difference between the two returns will be the equity premium. In this case, the return to capital is entirely determined by the linear technology, so the premium will be driven by adjusting the risk-free rate in accordance with the intertemporal marginal rate of substitution of households in the no-trade equilibrium, to which we turn next.

The intertemporal marginal rate of substitution (IMRS) of i-th household is given by

$$M_{i,t+1} = \beta \left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-1} \left(\frac{V_{i,t+1}}{E_t[V_{i,t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}$$

and includes the usual consumption growth term as well as deviation of the next-period value function from its certainty-equivalent that would capture news about future consumption. In the equilibrium, each household's IMRS is a valid stochastic discount factor, and so will be their cross-sectional average $M_{t+1} = \tilde{E}[M_{i,t+1}]$. Returns to capital and bonds must satisfy the following equations:

$$1 = E_t[M_{t+1}R^k_{t+1}],\ 1 = E_t[M_{t+1}R^b_{t+1}].$$

Assuming (conditional) lognormality, we can express the conditional equity premium in terms of logarithm of stochastic discount factor (SDF) and log returns as

$$E_t[r^k_{t+1}] + \frac{1}{2}\mathrm{Var}_t[r^k_{t+1}] - r^b_{t+1} = -\mathrm{Cov}_t[m_{t+1}, r^k_{t+1}]. \tag{1}$$

Since the capital return is exogenous, asset pricing properties will mainly depend on conditional distribution of the stochastic discount factor and its sensitivity to aggregate shocks.

To explicitly characterize the innovation to the logarithm of SDF, we need to find the innovation to the value function. To this purpose, define the logarithm of normalized value function $v_{i,t} = \log(V_{i,t}/C_{i,t})$ and rewrite the value function recursion as

$$\begin{aligned} v_{i,t} &= \beta\frac{1}{1-\gamma}\log E_t\left[\exp\left((1-\gamma)(v_{i,t+1} + \Delta c_{i,t+1})\right)\right] \\ &= \beta\frac{1}{1-\gamma}\log E_t\left[\exp\left((1-\gamma)(v_{i,t+1} + \Delta c_{t+1} - \frac{1}{2}\gamma x_{t+1})\right)\right], \end{aligned}$$

where the second line follows from substituting for individual consumption growth and integrating out idiosyncratic shock. Since the above expression involves only aggregate variables, clearly the normalized value function will be equalized across households: $v_{i,t} = v_t$. If we furthermore assume that a_t (and thus Δc_t) and x_t jointly follow Gaussian homoscedastic process, we get

$$v_t = \beta\left(E_t\left[v_{t+1} + \Delta c_{t+1} - \frac{1}{2}\gamma x_{t+1}\right] + \frac{1-\gamma}{2}\Sigma\right)$$

with $\Sigma = \mathrm{Var}_t\left[v_{t+1} + \Delta c_{t+1} - \frac{1}{2}\gamma x_{t+1}\right]$ being a (constant) conditional variance. Iterating forward and imposing proper terminal condition, value function can be expressed as

$$v_t = \frac{\beta}{1-\beta}\frac{1}{2}(1-\gamma)\Sigma + \sum_{i=1}^{\infty}\beta^i\left(\mathbb{E}_t\left[\Delta c_{t+i} - \frac{1}{2}\gamma x_{t+i}\right]\right).$$

The log of aggregate SDF in terms of v_{t+1} has the form of

$$m_{t+1} = \log(\beta) - \gamma\Delta c_{t+1} + (1-\gamma)(v_{t+1} - v_t/\beta) + \frac{1}{2}\gamma(1+\gamma)x_{t+1},$$

where the last term arises from integrating over cross-sectional consumption growth. The innovation to m_{t+1} can subsequently be shown to equal

$$m_{t+1} - E_t[m_{t+1}] = -\gamma\epsilon^c_{t+1} + \frac{1}{2}\gamma(1+\gamma)\epsilon^x_{t+1} - (\gamma-1)\eta^c_{t+1} + \frac{1}{2}\gamma(\gamma-1)\eta^x_{t+1}$$

where $\epsilon^c_{t+1} = \Delta c_{t+1} - E_t[\Delta c_{t+1}]$ is a short-run innovation to consumption growth, $\epsilon^x_{t+1} = x_{t+1} - E_t[x_{t+1}]$ is a short-run innovation to cross-sectional consumption growth variance, $\eta^c_{t+1} = (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \beta^i \Delta c_{t+1+j}\right]$ is an innovation to long-run expected consumption growth, and $\eta^x_{t+1} = (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \beta^i x_{t+1+j}\right]$ is an innovation to long-run expected cross-sectional variance. Increases in current or future consumption growth decrease marginal utility and thus carry a positive market price of risk, whereas increases in current or future cross-sectional variance enter with the opposite sign and thus carry a negative price of risk. In other words, assets which pay well in those states of the world in which a household receives bad news about current or *future* cross-sectional risk are less attractive and must offer higher returns.

In the above expression, the first term is standard and captures aggregate consumption growth. The second term is the same as in the Constantinides & Duffie model and captures contemporaneous effects of idiosyncratic risk. The third term describes news about future consumption, and has been studied in long run risk literature. The final term then captures news about future idiosyncratic risk, and is present only with preference for early resolution of uncertainty ($\gamma > 1$) and in a non-iid environment. The presence of this last term can potentially increase the equity premium if bad news about current and future consumption growth are accompanied by bad news about future levels of idiosyncratic risk.

As a more specific example, consider the following joint process for $\Delta c_t, x_t$:

$$\begin{aligned}\Delta c_t &= (1-\rho_c)\mu_c + \rho_c \Delta c_{t-1} + \epsilon_t, \ \ \epsilon_t \sim N(0, \sigma^2_\epsilon)\\ x_t &= \mu_x + \phi_x(\Delta c_t - \mu_c).\end{aligned}$$

so that aggregate consumption growth follows the AR(1) process and the idiosyncratic risk level is its affine function. Setting $\phi_x < 0$ corresponds to the countercyclical cross-sectional variance emphasized by Constantinides & Duffie. Since there is just one aggregate shock, we can obtain the following expression for log SDF innovation:

$$m_{t+1} - E_t[m_{t+1}] = \left(-\gamma + \frac{1}{2}\gamma(1+\gamma)\phi_x - (\gamma-1)\frac{\beta\rho_c}{1-\beta\rho_c} + \frac{1}{2}\gamma(\gamma-1)\phi_x\frac{\beta\rho_c}{1-\beta\rho_c}\right)\epsilon_{t+1}. \tag{2}$$

When $\gamma > 1$ and $\phi_x < 0$, all terms inside the paretheses have the same sign and their magnitude can be interpreted as the contribution of individual channels to the overall price of risk.

For a quantitative illustration, choose $\beta = 0.99$, $\gamma = 5$ (standard values), $\rho_c = 0.27$ (autocorrelation of quarterly US consumption growth) and $\phi_x = -0.16$ (see section

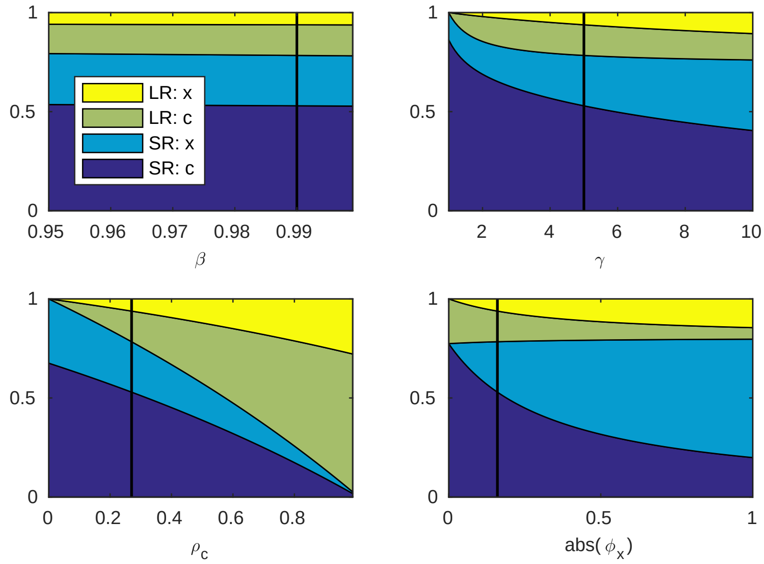

Figure 1: Comparative static of conditional Sharpe ratio decomposition according to equation (2). Filled areas show the relative contribution of each channel (long or short run, aggregate consumption or idiosyncratic risk). While varying each parameter, others are kept fixed ($\beta = 0.99, \gamma = 5, \rho_c = 0.27, \phi_x = -0.16$, see also black lines in corresponding subplots).

4.1). Following the above expression, we obtain that short-run consumption risk contributes 53.0%, short-run idiosyncratic risk 25.4%, long-run consumption risk contributes 15.5% and long-run idiosyncratic risk 6.2%. In relative terms, news about future idiosyncratic risk constitute 40% of the overall long-run risk. Figure 1 shows the sensitivity of this decomposition to each parameter. Varying the discount rate should in principle affect the weight households put on future events and thus also the relative importance of long run risk, but for the range of values usually considered it does not seem to play a large role. Higher risk aversion raises the share of both long run and idiosyncratic risk. Autocorrelation of consumption growth has a similar, although even stronger, effect, as with more predictability, a current shock to consumption causes greater revision of expectations about future. Finally, the degree of countercyclicality (plotted using its absolute value) makes the role of idiosyncratic risk larger.

The model presented in this section is too simplified in certain aspects. In a more standard production economy, the aggregate consumption process is endogenous and thus introduction of idiosyncratic risk may affect asset pricing results via general equilibrium effects. In addition, equity returns are also endogenous in the sense that the presence of idiosyncratic risk can affect the sensitivity of price-dividend ratios (and thus of returns themselves) to aggregate shocks, which might affect the predicted equity premium (although not the Sharpe ratio). For these reasons, in the next section I embed idiosyncratic risk into a version of a real business cycle model which will allow for both of these additional effects.

3 Full Model

This section describes the main model of a production economy with households facing idiosyncratic shocks. The model could be described as a variant of standard stochastic growth model, similar to Kaltenbrunner and Lochstoer (2010), modified with a tractable form of heterogeneity on the household side, modelled according to Constantinides and Duffie (1996).

3.1 Production

On the production side, there is a representative firm with standard Cobb-Douglas technology, producing output from capital K_t and labor H_t:

$$Y_t = K_t^{\alpha}(Z_t H_t)^{1-\alpha}, \tag{3}$$

where Z_t is labor-augmenting productivity and its log growth rate $\Delta z_t = \log(Z_t) - \log(Z_{t-1})$ is a given exogenous stochastic process. The firm hires labor on a competitive market at wage rate W_t to the point where wage equals the marginal product of labor:

$$W_t = (1-\alpha)\frac{Y_t}{H_t}. \tag{4}$$

The household labor supply is inelastic and fixed at unity, so in equilibrium

$$H_t = 1 \tag{5}$$

The firm owns its capital stock, uses part of its profits for investment I_t into the capital stock and pays the residual as dividend D_t:

$$Y_t = W_t H_t + I_t + D_t. \tag{6}$$

Capital accumulation is standard:

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{7}$$

Since the firm faces an intertemporal choice, it is necessary to discuss its objective. We shall assume the firm will choose an investment policy to maximize the present value of its dividends evaluated with a one-period stochastic discount factor M_{t+1} (to be discussed later), which is taken as given by the firm. Multi-period SDF is then defined as $M_{t\to t+j} = \prod_{i=1}^{j} M_{t+i}$, and the firm's objective is to maximize the sum of current dividend and (ex-dividend) stock price P_t^s, with the latter equal to the present discounted value of future dividends:

$$\max D_t + \underbrace{\mathbb{E}_t\left[\sum_{j=1}^{\infty} M_{t\to t+j} D_{t+j}\right]}_{P_t^s}.$$

Under constant returns to scale, return to the claim to firm's equity (priced with the SDF referred to above) will be equal to return on physical capital (Restoy and Rockinger 1994), in this case given by

$$R_{t+1}^K = \alpha\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta \tag{8}$$

and by standard variational arguments, firm's first order condition is

$$1 = \mathbb{E}_t \left[M_{t+1} R^K_{t+1} \right]. \tag{9}$$

Finally, resources left for aggregate consumption consist of wages and dividend payments, or, equivalently, of output less investment:

$$C_t = D_t + W_t H_t = Y_t - I_t. \tag{10}$$

Note that the production side of the model determines the dynamics of macroeconomic aggregates such as capital, output and consumption once the stochastic discount factor is specified. Of course, in equilibrium the SDF process captures the attitudes of households toward intertemporal choice and risk, so we shall discuss the household side of the model next.

3.2 Households

There is a continuum of households indexed by i, with each having (the same) Epstein-Zin preferences over its own consumption stream $\{C_{i,t}\}$, summarized by a recursion for the value function

$$V_{i,t} = \left\{ (1-\beta) C_{i,t}^{1-\rho} + \beta \mathbb{E}_t \left[V_{i,t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}, \tag{11}$$

where β captures time preference, ρ is the inverse of intertemporal elasticity of substitution and γ is relative risk aversion. Each household also inelastically supplies one unit of labor.

The main object of interest on the household side of the model is the stochastic discount factor, which enters into the firm's intertemporal decision. In a model with a representative household, we could drop the i subscript and the relevant SDF would be directly determined by the representative household's intertemporal marginal rate of substitution, the expression for which is known to be

$$M^{\mathrm{RA}}_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\mathbb{E}_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma}.$$

On the other hand, if households face idiosyncratic risks and markets are incomplete, so the risk cannot be insured away, we will observe dispersion in individual consumption growth rates. In principle, individual consumption is an endogenous outcome, depending on the household's optimal decisions, which are themselves functions of individual and aggregate state variables. Generally, the aggregate state would include a cross-sectional distribution of wealth, necessitating the use of complex solution methods, such as those used in Krusell and Smith (1998). Instead, I will follow Constantinides and Duffie (1996) and assume directly[5] that the resulting dispersion of consumption growth rates can be described by a multiplicative shock to the aggregate consumption growth:

$$\frac{C_{i,t+1}}{C_{i,t}} = \frac{C_{t+1}}{C_t} \exp(\eta_{i,t+1}) \tag{12}$$

where innovations $\eta_{i,t+1}$ are uncorrelated across households and across time. However, since we are interested in idiosyncratic risk with varying severity over the business cycle, we shall allow the distribution of $\eta_{i,t}$ to vary according to an exogenous parameter process x_t. It will turn out advantageous to summarize this dependence via a moment-generating function

$$G(\tau; x) = \mathbb{E}\left[e^{\tau\eta}|x\right] \tag{13}$$

and to assume that the parametrization satisfies the property $G(1, x) = 1$ for all possible x, ensuring that average consumption equals the aggregate consumption. For example, if $\eta_{i,t}$ is normal with variance x_t and mean $-x_t/2$, the MGF would be $G(\tau; x) = e^{(x/2)(\tau^2-\tau)}$.

The main advantage of the above approach is that it allows us to define the aggregate stochastic discount factor as a cross-sectional average of individual marginal rates of subtitution in a tractable way, so that the resulting expression depends only on aggregate variables. For this purpose, define the logarithm of value function scaled by individual consumption $v_{i,t} = \log(V_{i,t}/C_{i,t})$, as well as the logarithm of scaled certainty equivalent $\psi_{i,t} = \log\left(\mathbb{E}_t\left[V_{i,t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}/C_{i,t}\right)$, which satisfy the following:

$$\begin{aligned} v_{i,t} &= \frac{1}{1-\rho}\log\left((1-\beta) + \beta\exp((1-\rho)\psi_{i,t})\right) \\ \psi_{i,t} &= \frac{1}{1-\gamma}\log\left(\mathbb{E}_t\left[\exp((1-\gamma)(v_{i,t+1} + \Delta c_{i,t+1}))\right]\right) \end{aligned}$$

Under the maintained assumption on individual consumption growth, we have $\Delta c_{i,t+1} = \Delta c_t + \eta_{i,t+1}$, and the distribution of $\eta_{i,t+1}$ is the same for each household

[5] See section 3.4 for a discussion of how such a result could be derived as a particular equilibrium outcome.

from the point of view of period t. Using the law of iterated expectation to integrate over $\eta_{i,t+1}$ (conditional on the next-period parameters of its distribution x_{t+1}), we can rewrite the scaled value function recursion in terms of aggregates only, implying that these variables are equalized across households (thus we can drop the i subscript):

$$\begin{aligned} v_t &= \frac{1}{1-\rho} \log\left((1-\beta) + \beta \exp((1-\rho)\psi_t)\right) \\ \psi_t &= \frac{1}{1-\gamma} \log\left(\mathbb{E}_t\left[\exp((1-\gamma)(v_{t+1} + \Delta c_{t+1})) \cdot G(1-\gamma, x_{t+1})\right]\right) \end{aligned} \tag{14}$$

Note the MGF term $G(1-\gamma, x_{t+1}) = \mathbb{E}[\exp((1-\gamma)\eta_{i,t+1})|x_{t+1}]$, which arises from integrating over individual shock in the next period, conditional on its distribution which depends on aggregate variables x_{t+1}.

The individual household's intertemporal marginal rate of substitution is

$$M_{i,t+1} = \beta \left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-\rho} \left(\frac{V_{i,t+1}}{\mathbb{E}_t\left[V_{i,t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma} \tag{15}$$

which can be equivalently expressed as

$$M_{i,t+1} = \beta \exp\left(-\gamma \Delta c_{i,t+1} + (\rho-\gamma)(v_{t+1} - \psi_t)\right), \tag{16}$$

and subsequently the aggregate SDF is obtained by averaging over individual $M_{i,t+1}$ conditional on aggregate variables up to and including in period $t+1$:

$$M_{t+1} = \beta \exp\left(-\gamma \Delta c_{t+1} + (\rho-\gamma)(v_{t+1} - \psi_t)\right) \cdot G(-\gamma, x_{t+1}). \tag{17}$$

where again the term $G(-\gamma, x_{t+1})$ appears due to integration over individual shock.

Although defining aggregate SDF by averaging individual rates of substitution may seem arbitrary, if we grant that individual consumption allocations are outcomes of some (still unspecified) equilibrium, and abstracting from binding portfolio constraints, each household intertemporal rate of substitution would in fact be a valid SDF in the sense that it would be compatible with asset prices in the economy. Taking a cross-sectional average of these will result in a SDF which is valid too, but does not depend directly on any individual-level variables.

The presence of idiosyncratic risk thus affects the resulting discount factor through the properties of its distribution: specifically, through the $G(1-\gamma, x_{t+1})$ term in the value function recursion, provided that $\rho \neq \gamma$, as well as through $G(-\gamma, x_{t+1})$ term in the SDF. Since the modifications are expressed in terms of moment generating functions, all the higher moments of idiosyncratic risk could, in principle, affect the economy, although in the most commonly studied case of normal shocks, only the variance will matter. It is also clear that if the distribution of idiosyncratic shocks were time-invariant (i.e. x_t were constant), the only effect would be to introduce constant offsets into the value function and discount factor, while risk premia would not be affected directly. Finally, making the distribution of η collapse to a constant would yield expressions identical to those of a representative-agent version of the model, which can thus be considered a special case of the setup presented above.

3.3 Quantity dynamics and asset prices

To close the model, we need to further specify the exogenous process for productivity Z_t and the evolution of parameters x_t controlling the distribution of individual shocks (these could be functions of other aggregate variables, or follow their own exogenous process).

Productivity is assumed to be a random walk, so that

$$\Delta z_t = \mu_z + \sigma_z \epsilon_t, \epsilon_t \sim \mathcal{N}(0,1) \tag{18}$$

Regarding the form of individual risk, I will assume that the individual element of consumption growth is lognormal, so that

$$\eta_{i,t} \sim \mathcal{N}\left(-\frac{x_t}{2}, x_t\right)$$

and x_t representes its variance, which is exogenously given as an affine function of consumption growth

$$x_t = \mu_x + \phi_x(\Delta c_t - \mu_z). \tag{19}$$

Equations (18) and (19) together with equations (3), (5), (7), (8), (9), (10), (14), (17) and the functional form for $G(\tau, x)$ mean that we have a sufficient number of relationships for solving the model. Since there is no need to track cross-sectional distribution of assets, the model can be solved by standard perturbation methods after detrending.

In terms of asset prices, unlevered return to capital has been defined in (8), and its logarithm will be denoted $r^k_{t+1} = \log R^k_{t+1}$. We will define the price of a one-period riskless bond that pays one unit in the following period in a standard way:

$$P_t^b = \mathbb{E}_t \left[M_{t+1} \cdot 1 \right] \tag{20}$$

and define log-return on the bond as $r_{t+1}^b = \log(1/P_t^b)$. The excess return is the difference between return to capital and return to bonds: $r_{t+1}^x = r_{t+1}^k - r_{t+1}^b$. The conditional equity premium and Sharpe ratio are then defined as:

$$\begin{aligned} \mathrm{EP}_t &= \mathbb{E}_t[r_{t+1}^x] \\ \mathrm{SR}_t &= \frac{\mathbb{E}_t[r_{t+1}^x]}{\sqrt{\mathrm{Var}_t[r_{t+1}^x]}} \end{aligned} \tag{21}$$

and their unconditional averages are $\mathrm{EP} = \mathbb{E}[\mathrm{EP}_t]$, $\mathrm{SR} = \mathbb{E}[\mathrm{SR}_t]$.

Recall the expression for conditional equity premium in a lognormal setting (adjusted for Jensen inequality) from equation (1):

$$\mathbb{E}_t[r_{t+1}^k] + \frac{1}{2}\mathrm{Var}_t[r_{t+1}^k] - r_{t+1}^b = -\mathrm{Cov}_t[m_{t+1}, r_{t+1}^k].$$

In case of just one aggregate shock, so that $m_{t+1} - \mathbb{E}_t[m_{t+1}] = \eta_{m\epsilon}\epsilon_{t+1}$, the conditional Sharpe ratio and equity premium is approximately

$$\mathrm{SR}_t \approx |\eta_{m\epsilon}|\sigma_z, \;\; \mathrm{EP}_t = \mathrm{SR}_t \mathrm{Var}_t[r_{t+1}^x].$$

In the model, all conditional volatility of returns arises from fluctuations in the marginal product of capital, which is not volatile enough to match the observed variation in stock returns. This issue could in principle be fixed by introducing capital adjustment costs or leveraged equity, although in this paper I will focus mainly on the price rather than quantity of risk, i.e. on the Sharpe ratio.

3.4 No trade equilibrium

The model presented so far relies on a reduced-form way to incorporate idiosyncratic consumption risk. It is possible to support such an outcome as a no-trade equilibrium[6] of a model with households facing particularly defined idiosyncratic additive

[6]The discussion here adapts the no-trade equilibrium setup of Constantinides and Duffie (1996) from endowment to a production economy with EZ preferences. A close, although not identical aggregation approach is offered in Braun and Nakajima (2012), who allow for elastic labor supply, but also consider only time-separable utility function.

shocks to their budget constraints, which could represent unexpected expenditures, gains or redistitributive payments (which, however, cancel out in the aggregate) that cannot be insured against due to incomplete markets. Intuitively, given that a household's utility function is homothetic and in the proposed equilibrium the deviation of individual consumption from the aggregate is a geometric random walk with shocks uncorrelated in time, all the households behave essentially symmetrically in their consumption/saving and portfolio decisions, thus implying no trade in assets. No trade, together with symmetric initial portfolios, in turn lead to individual consumption heterogeneity of the form described in previous sections. For completeness, this section will present such an equilibrium in more detail.

The individual household receives labor income and can trade firm shares and bonds. Its budget constraint reads:

$$C_{i,t} + P_t^s A_{i,t+1} + P_t^b B_{i,t+1} = W_t + (P_t^s + D_t)A_{i,t} + B_{i,t} + \Upsilon_{i,t} C_t,$$

where P_t^s, P_t^b are prices of firm equity and a risk-free one-period bond respectively, $A_{i,t}, B_{i,t}$ are the household's beginning-of-period portfolio positions, and other variables are as defined previously. The household also faces an additive shock $\Upsilon_{i,t}$ to its wealth, scaled by the current level of aggregate consumption. We will require that the cross-sectional average of $\Upsilon_{i,t}$ equals zero, so that individual shocks do not add or subtract resources to the economy.

The evolution of idiosyncratic shock is specified as:

$$\Upsilon_{i,t} = (1 + \Upsilon_{i,t-1}) \exp(\eta_{i,t}) - 1$$

where $\eta_{i,t}$ are the same shocks which were previously characterized in equation (13). Since we assumed $\int \exp(\eta_{i,t}) \mathrm{d}i = 1$, the above law of motion maintains a zero cross-sectional mean of $\Upsilon_{i,t}$. For example, if $\eta_{i,t}$ is normally distributed, $\Upsilon_{i,t}$ will have a lognormal distribution shifted by a negative constant.

The household takes asset prices, wages, dividends, aggregate consumption and idiosyncratic shocks as given, and chooses its consumption and portfolio positions to maximize its value function (11). Given the allocation of consumption across households, the rest of the model functions as previously described, although we will also require that stock and bond prices are consistent with market clearing in financial markets, so that, in the aggregate, households own the whole firm ($\int A_{i,t} \mathrm{d}i = 1$) and bonds are in zero net supply ($\int B_{i,t} \mathrm{d}i = 0$). Given the specification of exogenous shocks $Z_t, \Upsilon_{i,t}$, the equilibrium of the economy can be thus defined as:

- stochastic process for aggregate output Y_t, consumption C_t, investment I_t, capital K_t, wage W_t, return to capital R_t^k and dividend D_t,
- firm equity price P_t^s and bond price P_t^b
- individual household consumption $C_{i,t}$, portfolio positions $A_{i,t}, B_{i,t}$, value function $V_{i,t}$ and IMRS $M_{i,t+1}$
- aggregate SDF M_{t+1}

such that

- given the aggregate SDF, $Y_t, I_t, K_t, C_t, D_t, R_t^k, W_t$ are consistent with firm optimality condition (9), production function (3), capital accumulation (7), resource constraints (6), (10) and marginal products (4), (8).
- markets for financial assets clear.
- $C_{i,t}$, $A_{i,t}, B_{i,t}$, $V_{i,t}$ and $M_{i,t+1}$ are consistent with optimal decisions by a household.
- M_t is consistent with cross-sectional aggregation of household intertemporal rates of substitution $M_{i,t}$ as described in (17).

Next, notice that if households held symmetric market-clearing portfolios, i.e. $\forall t, \forall i :$ $A_{i,t} = 1, B_{i,t} = 0$, their consumption growth would be in fact described by (12), since in such case their consumption is $C_{i,t} = W_t + D_t + \Upsilon_{i,t} C_t = (1 + \Upsilon_{i,t}) C_t$ and their consumption growth thus satisfies

$$\frac{C_{i,t+1}}{C_{i,t}} = \frac{C_{t+1}}{C_t} \frac{1 + \Upsilon_{i,t+1}}{1 + \Upsilon_{i,t}} = \frac{C_{t+1}}{C_t} \exp(\eta_{i,t+1})$$

The following result shows that an outcome where households hold symmetric portfolios at all times, embedded within the rest of the model described previously, is in fact an equilibrium:

Claim: Consider an allocation where

- firm stock price is given by $P_t^s = K_{t+1}$ and bond price is determined by aggregate SDF as in (20),
- households hold symmetric portfolios $A_{i,t} = 1, B_{i,t} = 0$,
- and rest of the model functions as described previously;

then such an allocation is an equilibrium. Moreover, households are in agreement in terms of the firm's investment policy.

To see why the above holds, we need to check whether first-order conditions of individual households are satisfied. The intertemporal rate of substitution of household i between two consecutive periods (implicitly, taking as given current aggregate state of the economy; I also supress time indices for clarity) can be generally written as a function of some first-period individual state s_i and second-period individual shock η_i' and aggregate shock ϵ': $M_i(s_i, \eta_i', \epsilon')$. In our case, however, individual IMRS given by (16) depends on the individual state only through the household's consumption growth, which is assumed to be uncorrelated over time and determined by future idiosyncratic shock η_i'. Therefore individual IMRS does not depend on the initial individual state and can be written as $M(\eta_i', \epsilon')$. Intuitively, if individual consumption behaves like a multiplicative random walk and households have homothetic preferences, any differences in wealth are simply a matter of scale.

The aggregate stochastic discount factor is obtained by averaging over individual shocks: $M(\epsilon') = \mathbb{E}\left[M_i(\eta_i', \epsilon') \mid \epsilon'\right]$ (since distribution of shocks is symmetric across households, this does not actually depend on i). We can then show that the aggregate optimality condition $\mathbb{E}\left[M(\epsilon')R(\epsilon')\right]$ for some return R also implies individual optimality $\mathbb{E}\left[M_i(\eta_i', \epsilon')R(\epsilon')\right]$, since here this follows directly from the law of iterated expectations. The aggregate optimality is satisfied by return on bonds by assumption, and it is easy to show that it also holds for return on stocks held by households[7]. It then follows that the household individual optimality conditions are also satisfied and that a no-trade euqilibrium is consistent with optimal consumption and portfolio choice by households.

The same argument also ensures that households do not differ in their preferred investment policy (see also Carceles-Poveda and Coen-Pirani (2009) for a more general discussion of when this is true): in equilibrium, each household receives the stream of dividends from the firm, so its preferred policy is to maximize the present value of future dividends, using its own IMRS as a discount factor. This would lead to a first order condition for investment $1 = \mathbb{E}[M_i(\eta_i', \epsilon')R^K(\epsilon')]$, but by the same logic of iterated expectations, this is equivalent to the assumed firm's condition (9).

[7] This can be verified by plugging in the proposed expression for stock price into the definition of return and using the fact that $D_{t+1} = Y_{t+1} - W_{t+1} - I_{t+1} = \alpha Y_{t+1} - I_{t+1}$. After some rearranging, we obtain that the stock return is equal to the return to capital defined in (8), and thus satisfies the condition due to the firm's optimality condition (9).

Another possible question is whether a different choice of weights across households when defining the aggregate SDF might affect the results. In general this is possible in models with incomplete markets (Carceles-Poveda 2009), but it turns out that in the current model weighting does not matter. Any weights corresponding to some reasonable corporate governance mechanism should depend only on current states of firm owners, not on realizations of next-period shocks. A weighted SDF $\tilde{M}(\epsilon') = \mathbb{E}\left[w(s)M_i(s, \eta_i', \epsilon') \mid \epsilon'\right]$ will not make a difference when M_i is independent of s.

4 Results

To evaluate how the addition of idiosyncratic risk affects the behavior of the neoclassical growth model, I first calibrate most of the parameters based on a representative-agent version of the model, then solve the model with and without idiosyncratic risk, and inspect its properties. In the second part of this section, I proceed by describing a log-linear approximate solution to the model, which is helpful to illustrate the interplay between idiosyncratic risk and dynamics of macroeconomic aggregates in the model. Finally, I will also consider an alternative way to model cyclical variation in the distribution of idiosyncratic risk by way of cyclical skewness rather than variance.

4.1 Calibration

Model calibration is summarized in table 1. Frequency is quarterly. Starting with a representative-agent version of the model, most parameters are chosen close to standard values in the literature, as in, e.g., Campbell (1994). α is set to match the capital share of income of one third, δ implies annual depreciation rate of 10%. Discount rate β and the inverse of IES ρ are set so as to match the steady state return to capital of 6% per annum and output growth being twice as volatile as consumption growth. Trend productivity growth is set at 2% per year. The volatility of productivity shocks matches standard deviation of quarterly output growth of 1%, roughly corresponding to postwar US data. Finally, risk aversion is set to 5, a relatively standard value.

Following Storesletten, Telmer, and Yaron (2007), who use a process for variance of

Parameter	Value	Description
β	0.988	discount factor
ρ	0.7	inverse of IES
γ	5	risk aversion
α	0.33	capital share
δ	0.025	depreciation rate
μ_z	0.005	mean productivity growth
σ_z	0.015	volatility of productivity shock
μ_x	0.0036	mean level of ind. risk
ϕ_x	-0.16	cyclicality of ind. risk

Table 1: Parameter values.

idiosyncratic shocks of the same form, I set $\mu_x = 0.0036$ (i.e. their value 0.014 rescaled to quarterly setting) and $\phi_x = -0.16$. The average level μ_x corresponds to annualized standard deviation of individual consumption growth of about 12%. The value of sensitivity ϕ_x captures the sensitivity of idiosyncratic risk to the business cycle, with negative values representing counter-cyclical variation. Given that quarterly (non-annualized) standard deviation of consumption growth will be approximately half a percent and assuming a normal distribution, the chosen value implies that fluctuations in x_t correspond to the annualized standard deviation of individual consumption growth ranging from approximately 9% to 15% with 95% probability (in terms of the ergodic distribution).

After detrending by productivity (a list of detrended equations can be found in the appendix), I solve the model by a 3rd-order perturbation method using Dynare (Adjemian et al. 2011), as higher-order approximation is necessary to obtain non-zero risk premia when the perturbation approach is used for numerical solution. Model-implied moments for various variables are then computed from a pruned representation of the system, using the approach and code presented by Andreasen, Fernandez-Villaverde, and Rubio-Ramirez (2013). In a recent work, Pohl, Schmedders, and Wilms (2018) argue that models with long-run risk can exhibit nonlinearities that make local approximations potentially unreliable, and suggest using global solution methods. It turns out that in the model presented here, nonlinearities are quite mild, so that local and global solutions yield very similar results, as documented in the appendix.

	data	model: RA	model: HA1	model: HA2
moments:				
$\sigma[\Delta y_t]$	1.90%	2.02%	2.01%	2.02%
$\sigma[\Delta c_t]/\sigma[\Delta y_t]$	0.56	0.50	0.74	0.49
$\sigma[\Delta i_t]/\sigma[\Delta y_t]$	2.58	2.65	1.81	2.63
$\text{cor}(\Delta y_t, \Delta y_{t-1})$	0.37	0.03	0.02	0.03
$\text{cor}(\Delta c_t, \Delta c_{t-1})$	0.27	0.21	0.06	0.21
Sharpe ratio	0.39	0.121	0.163	0.161
risk price decomposition:				
short run, Δc	-	39.1%	45.8%	29.7%
short run, x	-	0.0%	22.0%	14.2%
long run, Δc	-	60.9%	23.0%	40.1%
long run, x	-	0.0%	9.2%	16.0%

Table 2: Comparison of model-implied annualized moments. Data: US quarterly series 1947-2016; see appendix for definitions. Model RA: calibrated as in table 1, but setting $\mu_x = \phi_x = 0$. Model HA1: as in table 1, but setting $\beta = 0.973$ to match RA model steady state. Model HA2: as in table 1, but setting $\beta = 0.975, \rho = 0.214$ to match RA model steady state and quantity dynamics. Standard deviations and Sharpe ratio are annualized by doubling from quarterly values. The bottom section shows relative contributions to the price of risk based on loglinear approximation.

4.2 Quantitative results

Table 2 displays selected unconditional moments from three versions of the model, as well as from US quarterly macroeconomic data. A representative agent variant of the model (RA column) matches variances of output and consumption growth (which, of course, it has been calibrated to match), as well as autocorrelation of consumption growth. The implied Sharpe ratio of about 12% is lower than observed, yet still quite substantial compared to its value in a model with separable utility (approximately 0.6%). The second variant (HA1 column) is a model with idiosyncratic risk parameters calibrated as described above and otherwise the same as a representative-agent model, with the exception of the discount factor β which has been adjusted to obtain the same steady state. Looking at our main object of interest, we see that the presence of countercyclical idiosyncratic risk has increased the market price of risk (proxied here

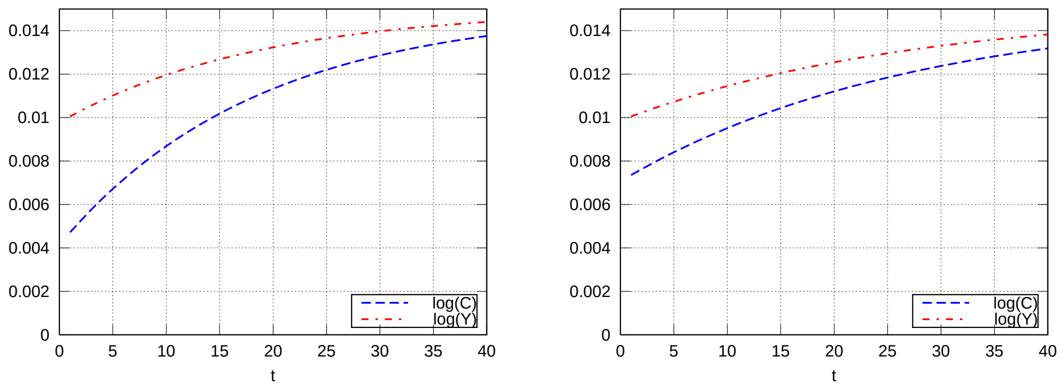

Figure 2: Impulse responses of log consumption and output to a 1 s. d. (permanent) productivity shock. Right panel: models RA (representative agent) and HA2 (het. agents, with β and ρ adjusted to match RA model dynamics). Left panel: model HA1 (het. agents, with β adjusted to match RA model steady state).

by the Sharpe ratio of excess returns) by approximately a third, but the dynamics of macroeconomic quantities has also changed significantly: with idiosyncratic risk, aggregate consumption growth has volatility closer to that of the output growth and autocorrelation closer to zero, which worsens the empirical fit of the model. In the third version (HA2 column), both the discount factor and the intertemporal elasticity of substitution are modified to maintain the same dynamics of output and consumption as in the RA variant of the model. We can see that the market price of risk remains high, so that by using a suitable choice of preference parameters, the model can be relatively succesful along both dimensions.

The bottom part of the table presents decomposition of the risk premium based on loglinear approximation, similar to the discussion in section 2 (see also the next subsection and the appendix for more details about loglinear solution). Dispersion of idiosyncratic shocks constitutes a bit less than a third of the overall long run risk contribution and around a third of the overall short run risk contribution. The overall contribution of long run risk is 61% in representative agent model and 56% in the HA2 model, but it is only 32% in the HA1 model, due to the overall amount of predictability in the economy being lower (the aggregate consumption is closer to a random walk).

To better understand how the introduction of idiosyncratic risk affects the behavior of output and consumption, figure 2 plots impulse responses to a productivity shock of output and consumption (log) levels for both RA and HA1 variants of the model (impulse responses in HA2 calibration are by construction close to the RA variant). The representative agent version shows both consumption and output growing over time toward their new, permanently higher, values implied by the permanent increase in productivity, but the response of consumption on impact is about half of output response (in line with calibration targeting volatility of consumption growth being half of output growth volatility). Thus households are willing to spread consumption increases over a longer horizon and to accept variation in future consumption growth rates in order to accumulate capital stock more quickly and thus to obtain more benefits from the increased productivity. However, in the model with idiosyncratic risk, the response of consumption on impact is much stronger and essentially consumes the whole productivity gain straight away at the cost of slower accumulation of capital, as if households were much more averse to intertemporal substitution of consumption.

This effect on consumption smoothing also complicates the analysis of asset prices, since the price of risk can be affected by the presence of idiosyncratic risk, in addition to its direct impact on the stochastic discount factor described in section 2, also through the changes in the endogenous process for aggregate consumption caused by a lower steady state interest rate and lower "aggregate" intertemporal elasticity of substitution. Specifically, with less predictable consumption growth, the long run consumption risk emphasized by Kaltenbrunner and Lochstoer (2010) becomes less important, although the overall market price of risk has gone up in our case. On the other hand, as can be seen from the final column of table 2, it is possible to counteract such impacts by increasing IES (i.e. decreasing ρ) of individual households, although in general the size of the adjustment will depend on both the level and cyclicality of idiosyncratic risk, as well as households risk aversion, as discussed in more detail in the next subsection.

4.3 Qualitative analysis

To gain better intuition about the implications of idiosyncratic risk, we shall inspect a loglinear approximation to the model solution along the lines of Campbell (1994). Since the productivity process is a random walk, the detrended model has just one relevant state variable, (log) ratio of capital and productivity $k_t^* = \log(K_t/Z_t)$ (in terms of notation, lowercase symbols shall denote logs and starred variables are detrended by productivity). The dynamics of capital, output and consumption are determined by the deterministic steady state and by the sensitivity of detrended consumption to detrended capital: $\tilde{c}^*_t = \eta_{ck}\tilde{k}^*_t$, with a tilde denoting deviation from the steady state value.

A complete derivation can be found in the appendix, but it is possible to show that the steady state depends on preference and idiosyncratic risk parameters only through their effect on steady state return to capital $\bar{r}^k = -\log(\beta) + \rho\mu_z - \frac{1}{2}\gamma(1+\rho)\mu_x$. The coefficient η_{ck} depends on the steady state, as well as on the "effective" inverse of IES $\hat{\rho} = \rho - \frac{1}{2}\gamma(1+\rho)\phi_x$. In other words, any combinations of parameters $\beta, \rho, \gamma, \mu_x, \phi_x$ which imply the same $\hat{r}^k$ and $\hat{\rho}$ will lead to identical dynamics of output and consumption growth.

More specifically, if we start with a representative-agent model with parameters $\beta^{RA}, \rho^{RA}, \gamma^{RA}$ (i.e. $\mu_x^{RA} = \phi_x^{RA} = 0$), and then introduce idiosyncratic risk by setting $\mu_x > 0, \phi_x \neq 0$, we can maintain the same quantity dynamics in the heterogeneous-agent model by choosing parameters $\beta^{HA}, \rho^{HA}, \gamma^{HA}$ such that

$$-\log(\beta^{RA}) + \rho^{RA}\mu_z = -\log(\beta^{HA}) + \rho^{HA}\mu_z - \frac{1}{2}\gamma^{HA}(1+\rho^{HA})\mu_x$$
$$\rho^{RA} = \rho^{HA} - \frac{1}{2}\gamma^{HA}(1+\rho^{HA})\phi_x$$

If we, for example, decide to keep risk aversion the same: $\gamma^{HA} = \gamma^{RA}$, the above two equations pin down the new values of the discount rate and intertemporal elasticity of substitution. If the individual risk was acyclical ($\phi_x = 0$), the only necessary adjustment is in the discount rate, which should be set lower to counteract the precautionary saving effect pushing interest rates down. In the presence of countercyclical individual risk ($\phi_x < 0$), we would additionally need to make ρ^{HA} lower[8], to counteract the greater aversion of agents to intertemporal substitution.

Why do agents exhibit this aversion? We can gain some intuition by looking at the power utility case ($\gamma = \rho$). The individual Euler's equation can be then written approximately as

$$\log(\beta) + \rho E_t[\Delta c_{i,t+1}] - \frac{1}{2}\rho^2 Var_t[\Delta c_{i,t+1}] = r^b_{t+1}$$

Since $\Delta c_{i,t+1} = \Delta c_{t+1} + \eta_{i,t+1}$, if we ignore the small normalization shift in $\eta_{i,t+1}$, expected individual consumption growth moves one to one with aggregate expected

[8] A similar expression for "effective" intertemporal substitution in CRRA case was derived in Constantinides and Duffie (1996).

consumption growth. However, with countercyclical risk, the conditional variance of individual consumption growth will vary inversely to Δc_{t+1}, and thus the whole left hand side will be more sensitive to $E_t[\Delta c_{t+1}]$. As a result, if we considered only aggregate data, the agent behaves as if he had higher ρ (lower intertermporal substitution) than he really does, which is consistent with empirical estimates of IES finding higher values when estimated on micro data compared with findings from aggregate time series (Havranek 2015).

Moreover, if the agent has Epstein-Zin preferences with risk aversion differing from the inverse of IES, the above result suggests that the degree of required adjustment in ρ depends on risk aversion as well, or alternatively, that risk aversion affects the dynamics of macroeconomic aggregates even at a first order approximation. The separation property described by Tallarini (2000) (i.e. that risk aversion affects the risk premia but not the behavior of quantities) thus does not hold outside the representative-agent model. A related issue with the proposed adjustment might be that, if idiosyncratic risk is strongly cyclical (ϕ_x has large magnitude) or households are very risk averse (γ is high), the adjustment might imply parameter values for ρ that are too low or even negative. It is possible that introducing other extensions affecting intertemporal choice, such as habit formation, might counteract this tendency, although I do not follow this direction in the current paper.

Even though the above discussion would suggest that the effect of idiosyncratic risk (at least as modelled here) does not affect qualitative properties of the representative-agent model conditional on suitable recalibration of preference parameters, the equivalence does not carry over to asset prices. Up to a linear approximation, log of scaled value function $v_t = \log(V_{i,t}/C_{i,t})$ can also be solved for as a function of capital stock, so that in terms of deviations from steady state, $\tilde{v}_t = \eta_{vk}\tilde{k}^*_t$. The coefficient η_{vk} is a function of the steady state and η_{ck}, but depends also on both μ_x and ϕ_x. With countercyclical risk ($\phi_x < 0$), the value function will be more sensitive to detrended capital stock and thus also to a productivity shock. The innovation to log SDF can be written as

$$m_{t+1} - E_t[m_{t+1}] = \underbrace{-\left[\left(\gamma - \frac{1}{2}\gamma(1+\gamma)\phi_x\right)\eta_{cz} + (\gamma-\rho)(-\eta_{vk})\right]}_{\eta_{m\epsilon}}\epsilon_{t+1} = \eta_{m\epsilon}\epsilon_{t+1}$$

implying a conditional Sharpe ratio

$$\frac{\log\left(E_t[R^k_{t+1}]\right) - r^b_{t+1}}{sd_t[r^k_{t+1}]} = -\eta_{m\epsilon}\sigma_z$$

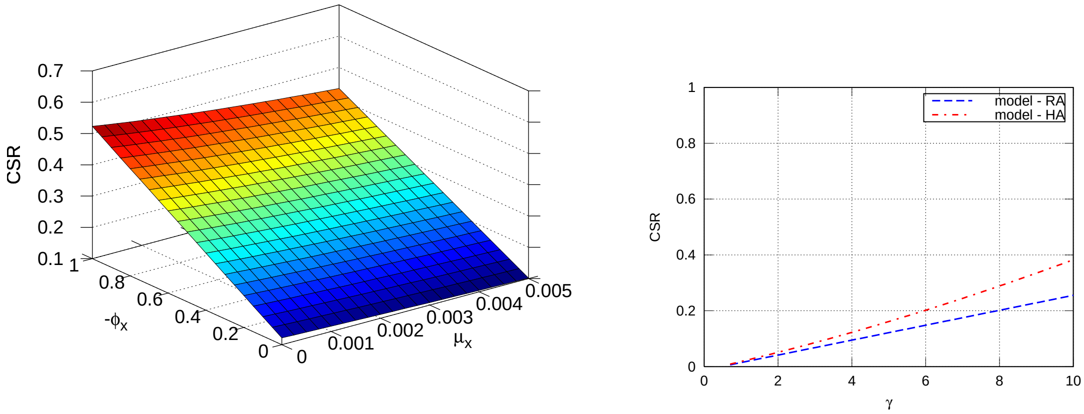

Figure 3: Comparative static for the conditional Sharpe ratio. Left: dependence on idiosyncratic risk parameters. Right: dependence on risk aversion. At each point, ρ and β are recalibrated to imply the same dynamics of aggregate quantities.

Therefore, even if we recalibrate the parameters to maintain the same dynamics of aggregate consumption, market price of risk will still differ from the one implied by the representative-agent model with the same dynamics.

The left panel of figure 3 plots the (annualized) conditional Sharpe ratio as a function of μ_x, ϕ_x when preference parameters are recalibrated to match the quantity dynamics of the representative-agent model solved previously. Each point on the graph thus implies the same consumption process so that we can distinguish the pure effects of idiosyncratic risk on the risk premium. If the risk was acyclical ($\phi_x = 0$), the price of risk would actually go slightly down due to lower required discount rate, which in turn weakens the impact of long-run consumption risk (this effect is present only when consumption growth is not iid, otherwise acyclical idiosyncratic risk would have no impact, as in Krueger and Lustig (2010)). However, making the risk countercyclical increases the price of risk substantially. Note that Epstein-Zin preferences are crucial for this result, since if we imposed $\gamma = \rho$, we would obtain $\eta_{m\epsilon} = -\hat{\rho}\eta_{cz}$ and thus the recalibration procedure would imply the same price of risk for any combination of parameters.

The right panel of figure 3 plots the dependence of the risk premium on the risk aversion parameter, for a representative-agent model and for a model with idiosyncratic risk calibrated as in the previous section, again while keeping the quantity dynamics the same. We can observe that the presence of idiosyncratic risk not only makes the risk premium rise faster with higher risk aversion, but it causes it to do so at an increasing rate, leading to a convex relationship (whereas the dependence is linear in RA model). This confirms that the combination of Epstein-Zin preferences with idiosyncratic risk leads to an interaction that makes it easier to match observed risk premia with lower levels of risk aversion.

4.4 Cyclical skewness

Recent research (Guvenen, Ozkan, and Song 2014) suggests that it is cyclical variation in skewness, rather than variance of idiosyncratic shocks that is more consistent with data. Although cyclical variance, as analyzed in the previous sections, is especially tractable given the loglinear form of moment generating function for Gaussian distribution, the model allows the use of other distributions as well, as long as their moment generating function can be expressed in closed form. To see how much the results described above depend on specific form of idiosyncratic risk, I solve the model with $\eta_{i,t}$ following a mixture of three normal distributions with time varying means, as proposed by McKay (2017)[9]. Specifically I assume that

$$\eta_{i,t} \sim \text{constant} + \begin{cases} N(\mu_{1,t}, \sigma_1^2) & \text{with prob. } p_1 \\ N(\mu_{2,t}, \sigma_2^2) & \text{with prob. } p_2 \\ N(\mu_{3,t}, \sigma_3^2) & \text{with prob. } p_3 \end{cases}$$

where the constant captures normalization, so that $E[\exp(\eta_{i,t})] = 1$, the means are given by

$$\begin{aligned} \mu_{1,t} &= 0 \\ \mu_{2,t} &= \mu_2 - x_t, \ \mu_2 < 0 \\ \mu_{3,t} &= \mu_3 - x_t, \ \mu_3 > 0 \end{aligned}$$

and, as before, x_t is a function of aggregate consumption growth:

$$x_t = \phi_x(\Delta c_t - \mu_z).$$

Individual consumption growth can belong either to the first mixture component, which stands for the "normal" experience faced by a majority of households, or to one

[9] To be precise, I use the distribution of the permanent component of income shock faced by employed agents in the model described in that paper.

of the other two components which represent negative or positive jumps. Movements in x_t then shift the position of the second and third components relative to first one, making the size of negative jumps larger during recessions (provided $\phi_x < 0$) and thus making the cross-sectional distribution of consumption growth more negatively skewed.

The calibration of means, variances and probabilities of the mixture elements follows McKay (2017), although I scale the overall size of the shock (i.e. means and standard deviations of mixture components) by one half to achieve a variance comparable to lognormal calibration used in previous sections. Sensitivity of x_t is estimated by regressing the time series for x_t provided by Alisdair McKay on his website[10] on US consumption growth, and the resulting coefficient is also scaled by one half. The chosen parameters are thus: $\mu_2 = -0.835$, $\mu_3 = 0.1970$, $\sigma_1 = 0.0319$, $\sigma_2 = \sigma_3 = 0.1668$, $p_1 = 94.87\%$, $p_2 = 3.24\%$, $p_3 = 1.89\%$ and $\phi_x = -7.285$. At the steady state, standard deviation of η with given parameters is 6.1%, or around 12.2% annualized, while the coefficient of skewness is 1.05 and of kurtosis 27.6, so the distribution is slightly positively skewed and fat-tailed. Measured in terms of plus/minus two standard deviations of aggregate consumption growth, skewness ranges from -1.5 to 3.1 over the business cycle.

Table 3, organized similarly as table 2, contains unconditional moments from two versions of a model with cyclical skewness. Again, I compare a version of the model with β recalibrated to match steady state return to capital (HA3 column), and another (HA4 column) with β and ρ recalibrated to match the dynamics of output and consumption[11]. The results are largely comparable to those in table 2, although the Sharpe ratio of 18% under skewed idiosyncratic shocks is somewhat higher compared to 16% under lognormal shocks. Without adjusting individual intertemporal elasticity of substitution, we again observe a change in the behavior of aggregate consumption, although the change is not as strong as in the lognormal case. Decomposition of risk premium is qualitatively also similar to the lognormal case, but quantitatively the role of idiosyncratic risk is slightly higher in relative terms.

[10] `http://people.bu.edu/amckay/files/risk_time_series.csv`

[11] It is possible to derive approximate formulas for adjusting the parameters as in the previous section, although they are somewhat more involved due to the necessity of loglinearizing MGF terms. However, qualitatively the direction of adjustment is same as before.

	data	model: RA	model: HA3	model: HA4
moments:				
$\sigma[\Delta y_t]$	1.90%	2.02%	2.01%	2.02%
$\sigma[\Delta c_t]/\sigma[\Delta y_t]$	0.56	0.50	0.76	0.53
$\sigma[\Delta i_t]/\sigma[\Delta y_t]$	2.58	2.65	1.69	2.47
$\text{corr}(\Delta y_t, \Delta y_{t-1})$	0.37	0.03	0.02	0.03
$\text{corr}(\Delta c_t, \Delta c_{t-1})$	0.27	0.21	0.05	0.17
SR	0.39	0.121	0.189	0.184
risk price decomposition:				
short run, Δc	-	39.1%	41.1%	26.4%
short run, x	-	0.0%	26.1%	16.8%
long run, Δc	-	60.9%	20.8%	36.1%
long run, x	-	0.0%	12.0%	20.8%

Table 3: Comparison of model-implied annualized moments under cyclical skewness. Data: US quarterly series 1947-2016; see the appendix for definitions. Model RA: calibrated as in table 1 without idiosyncratic risk. Model HA3: as in table 1 and section 4.4, but setting $\beta = 0.974$ to match RA model steady state. Model HA4: as in table 1 and section 4.4, but setting $\beta = 0.974, \rho = 0.255$ to match RA model steady state and quantity dynamics. Standard deviations and Sharpe ratio are annualized by doubling from quarterly values. The bottom section shows relative contributions to the price of risk computed using the loglinear approximation.

5 Conclusion

In this paper, I have studied how preferences for early resolution of uncertainty and idiosyncratic, uninsurable risk affect risk premia in a tractable macroeconomic model with production. On one hand, the combination of the two elements implies that households care about direct shocks as well as news about both aggregate consumption and the amount or shape of individual risk, and if the latter varies cyclicaly over time, both can increase the price of risk more than each element would in isolation. On the other hand, when households can shift consumption intertemporally by investing in productive capital, countercyclical risk affects their incentive to do so, and on the aggregate level, the economy behaves as if households had lower intertemporal elasticity of substitution, potentially leading to different behavior of macroeconomic quantities. Nevertheless, at least in the setting analyzed here, one can maintain the same quantity dynamics by suitably recalibrating preference parameters. Specifically, if we are willing to assume that individual agents have higher intertemporal elasticity of substitution, it is possible to compensate for the effect of cyclical risk on aggregate consumption while keeping the price of risk higher.

There are several directions that could be pursued in further research. Introducing elastic labor supply or habit formation would allow for greater flexibility in matching macroeconomic dynamics. It might be also interesting to investigate independent shocks to the process describing distribution of idiosyncratic risk, either as a source of macroeconomic fluctuations or as an asset pricing factor, although identifying such shocks might present a challenge. An additional direction to consider would be to include stochastic volatility of aggregate shocks, which is another channel of time-varying uncertainty often analyzed in the literature, in order to compare and contrast the effects of “macro” and “micro” uncertainty on the economy. Finally, closer comparison to models with more realistic structure of household heterogeneity and trade between households would be useful in establishing the validity of the modelling approach used in the present paper.

A Appendix

A.1 Detrended model equations

Notation:

Lowercase variable names usually denote logarithms, e.g. $k_t = \log(K_t)$. Starred variables denote variables detrended by productivity, i.e. $y_t^* = \log(Y_t/Z_t) = y_t - z_t$. Delta denotes 1st difference, e.g. $\Delta c_t = c_t - c_{t-1}$.

List of variables:

Variable	Description
Δz_t	productivity growth rate
y_t^*	log detrendend output
k_t^*	log detrended capital
c_t^*	log detrended agg. consumption
Δc_t	growth rates of output, consumption
r_t^k	log return to capital
p_t^b	log bond price
r_t^b	log return to risk-free bond
m_t	log of aggregate SDF
v_t	log of scaled value function
ψ_t	log of scaled certainty equivalent
x_t	variance of individual consumption growth rates
ϵ_t	productivity shock

Equations:

- The production block contains equations describing productivity growth, the production function, capital accumulation, marginal product of capital, the

Euler equation for investment and definition of consumption growth:

$$\begin{aligned}
\Delta z_t &= \mu_z + \epsilon_t \\
y_t^* &= \alpha k_t^* \\
\exp\left(k_{t+1}^* + \Delta z_{t+1}\right) &= (1-\delta)\exp\left(k_t^*\right) + \exp\left(y_t^*\right) - \exp\left(c_t^*\right) \\
\exp\left(r_t^K\right) &= \alpha \exp\left((\alpha-1)k_t^*\right) + 1 - \delta \\
1 &= E_t\left[\exp\left(m_{t+1} + r_{t+1}^K\right)\right] \\
\Delta c_{t+1} &= c_{t+1}^* - c_t^* + \Delta z_{t+1}
\end{aligned}$$

- The household block contains equations describing scaled value function, certainty equivalent, process of variance of individual consumption growth rates and the stochastic discount factor:

$$\begin{aligned}
v_t &= \frac{1}{1-\rho}\log\left(1-\beta+\beta\exp((1-\rho)\psi_t)\right) \\
\exp((1-\gamma)\psi_t) &= E_t\left[\exp((1-\gamma)(v_{t+1} + \Delta c_{t+1} - (\gamma/2)x_{t+1}))\right] \\
x_{t+1} &= \mu_x + \phi_x(\Delta c_{t+1} - \mu_z) \\
m_{t+1} &= \log(\beta) - \rho\Delta c_{t+1} + (\rho-\gamma)(v_{t+1} - \psi_t + \Delta c_{t+1}) + (1/2)\gamma(1+\gamma)x_{t+1}
\end{aligned}$$

- The remaining equations describe price and return of the risk-free bond:

$$\begin{aligned}
\exp(p_t^b) &= E_t\left[\exp\left(m_{t+1}\right)\right] \\
r_t^b &= -p_{t-1}^b
\end{aligned}$$

Steady state:

Setting productivity shocks to zero allows us to find a stationary steady state, which corresponds to the balanced growth path in terms of original, undetrended variables. We shall denote steady state values by dropping the time index and bars over the variables.

- Along the balanced growth path, productivity and consumption grow at the same rate, so $\overline{\Delta z} = \overline{\Delta c} = \mu_z$. Idiosyncratic risk is at its average level: $\bar{x} = \mu_x$.
- Given the constant consumption growth, we can solve for the value function

and steady state SDF:

$$\bar{v} = \frac{1}{1-\rho} \log \left(\frac{1-\beta}{1-\beta e^{(1-\rho)(\mu_z - (\gamma/2)\mu_x)}} \right)$$
$$\bar{\psi} = \bar{v} + \mu_z - (\gamma/2)\mu_x$$
$$\bar{m} = \log(\beta) - \rho\mu_z + \frac{1}{2}\gamma(1+\rho)\mu_x$$

- Steady state SDF determines the return to capital, which in turn allows us to solve for steady state capital, output and consumption:

$$\bar{r}^k = -\log(\beta) + \rho\mu_z - \frac{1}{2}\gamma(1+\rho)\mu_x$$
$$\bar{k}^* = \frac{1}{\alpha - 1} \log \left(\frac{\exp\left(\bar{r}^k\right) - 1 + \delta}{\alpha} \right)$$
$$\bar{y}^* = \alpha \bar{k}^*$$
$$\bar{c}^* = \log\left(\exp(\bar{y}^*) - (\exp(\mu_z) - 1 + \delta)\exp(\bar{k}^*)\right)$$

- Finally, the SDF determines the bond price and return, which equals the return to capital:

$$\bar{p}^b = \log(\beta) - \rho\mu_z + \frac{1}{2}\gamma(1+\rho)\mu_x$$
$$\bar{r}^b = -\log(\beta) + \rho\mu_z - \frac{1}{2}\gamma(1+\rho)\mu_x$$

A.2 Local vs. global solution

To find whether solving the model numerically with perturbation omits any substantial nonlinearities, I also solve a version of the model with counter cyclical variance also by using a projection method. I approximate consumption and value functions as combinations of Chebyshev polynomials up to the 10-th degree and solve for polynomial coefficients such that forward-looking conditions (i.e. the definition of the value function and the Euler equation, with expectations evaluated by 5-point Gauss-Hermite quadrature) hold exactly at a set of corresponding collocation nodes. Table 4 shows the resulting Sharpe ratios (obtained as averages from a simulation with each solution), which are very similar. Other moments are omitted as they were virtually identical up to 3 decimal places. Thus it seems that for the model and calibration studied here, nonlinearities do not matter very much.

	model: RA	model: HA1	model: HA2
3rd order perturbation			
Sharpe ratio	0.121	0.163	0.161
projection			
Sharpe ratio	0.119	0.160	0.160

Table 4: Comparison of solutions from perturbation and projection methods.

A.3 Linearized solution

The model summarized above has a single state variable, detrended capital k_t^* and thus its inearized solution can be found explicitly. We shall denote deviations from a steady state value by tilde, e.g. $\tilde{k}_t^* = k_t^* - \bar{k}^*$. First, linearize key equations around the steady state:

$$\begin{aligned}
\tilde{k}_{t+1}^* &= \lambda_1 \tilde{k}_t^* - \lambda_2 \tilde{c}_t^* - \epsilon_{t+1} \\
\tilde{r}_t^K &= \lambda_3 \tilde{k}_t^* \\
E_t[\tilde{r}_{t+1}^K] &= -E_t[\tilde{m}_{t+1}] \\
\tilde{m}_{t+1} &= -\gamma \widetilde{\Delta c}_{t+1} + (\rho - \gamma)(\tilde{v}_{t+1} - \tilde{\psi}_t) + (1/2)\gamma(1+\gamma)\tilde{x}_{t+1} \\
\tilde{v}_t &= \kappa \tilde{\psi}_t \\
\tilde{\psi}_t &= E_t\left[\tilde{v}_{t+1} + \widetilde{\Delta c}_{t+1} - (\gamma/2)\tilde{x}_{t+1}\right] \\
\widetilde{\Delta c}_{t+1} &= \tilde{c}_{t+1}^* - \tilde{c}_t^* + \widetilde{\Delta z}_{t+1} \\
\widetilde{\Delta z}_{t+1} &= \epsilon_{t+1} \\
\tilde{x}_{t+1} &= \phi_x \widetilde{\Delta c}_{t+1}
\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ and κ are defined as

$$\begin{aligned}
\lambda_1 &= \exp\left(\bar{r}^k - \mu_z\right) \\
\lambda_2 &= \exp\left(\bar{c}^* - \bar{k}^* - \mu_z\right) \\
\lambda_3 &= \alpha(\alpha - 1)\exp\left((\alpha - 1)\bar{k}^* - \bar{r}^K\right) \\
\kappa &= \beta \exp((1-\rho)(\mu_z - (\gamma/2)\mu_x))
\end{aligned}$$

We are looking for consumption policy in the form of $\tilde{c}_t^* = \eta_{ck}\tilde{k}_t^*$.

Claim: if we can write the expected log SDF as $E_t[\tilde{m}_{t+1}] = -\hat{\rho}E_t\left[\widetilde{\Delta c}_{t+1}\right]$ for some $\hat{\rho}$, then η_{ck} can be found by using the method of undetermined coefficients as a (positive) solution to the quadratics

$$\hat{\rho}\lambda_2\eta_{ck}^2 + (\hat{\rho} - \lambda_2\lambda_3 - \hat{\rho}\lambda_1)\,\eta_{ck} + \lambda_1\lambda_3 = 0.$$

Proof: substitute law of motion for capital and consumption policy into the linearized Euler equation, take expectation (simply cancels shock), rearrange. There will be two real roots, one positive, one negative (since $\hat{\rho}\lambda_2 > 0$ and $\lambda_1\lambda_3 < 0$), and the positive one corresponds to the stable solution. $\square$

Claim: our model satisfies the above with

$$\hat{\rho} = \rho - \frac{1}{2}\gamma(1+\rho)\phi_x.$$

Proof: since

$$\tilde{v}_{t+1} - \tilde{\psi}_t = \tilde{v}_{t+1} - E_t[\tilde{v}_{t+1}] - E_t[\widetilde{\Delta c}_{t+1}] + (\gamma/2)E_t[\tilde{x}_{t+1}]$$

and

$$E_t[\tilde{v}_{t+1} - \tilde{\psi}_t] = -E_t[\widetilde{\Delta c}_{t+1}] + (\gamma/2)E_t[\tilde{x}_{t+1}]$$

after bit of algebra, we get

$$E_t[\tilde{m}_{t+1}] = -\left(\rho - \frac{1}{2}\gamma(1+\rho)\phi_x\right)E_t\left[\widetilde{\Delta c}_{t+1}\right]$$

$\square$

Finally, we can also solve for the value function in the form of $\tilde{v}_t = \eta_{vk}\tilde{k}_t^*$, also by using the method of undetermined coefficients. The result:

$$\eta_{vk} = \frac{\kappa\left(1 - \frac{\gamma}{2}\phi_x\right)\eta_{ck}\left(\lambda_1 - \lambda_2\eta_{ck} - 1\right)}{1 - \kappa\left(\lambda_1 - \lambda_2\eta_{ck}\right)}.$$

Having solved for the consumption and value functions, innovation to the log SDF can be expressed as

$$m_{t+1} - E_t[m_{t+1}] = \left(\gamma(1-\eta_{ck}) + (\gamma-\rho)(-\eta_{vk}) + \frac{1}{2}\gamma(1+\gamma)(-\phi_x)(1-\eta_{ck})\right)(-\epsilon_t)$$

Since typically $\gamma > \rho$, $\eta_{vk} < 0$ and $\phi_x < 0$, each of the three added terms inside the large parentheses is positive and can be understood as standing for short-run aggregate consumption risk, long run risk and short-run idiosyncratic risk, respectively. To further decompose long run risk, iterate forward on the definition of $\tilde{v}_t$ to obtain

$$\tilde{v}_t = \sum_{i=1}^{\infty}\kappa^i\left(E_t[\widetilde{\Delta c}_{t+i}] - \frac{1}{2}\gamma E_t[\tilde{x}_{t+i}]\right) = \left(1 + \frac{1}{2}\gamma(-\phi_x)\right)\sum_{i=1}^{\infty}\kappa^i E_t[\widetilde{\Delta c}_{t+i}]$$

so that the share of long run risk attributable to news about x can be taken as $\frac{\frac{1}{2}\gamma(-\phi_x)}{\left(1+\frac{1}{2}\gamma(-\phi_x)\right)}$.

A.4 Linearized solution with general MGF

The previous derivation of loglinear approximation can be relatively easily extended to the case of a general moment-generating function describing the distribution of idiosyncratic shocks. Specifically, let $G(t,x)$ be the MGF as described in the main text (normalized so that $G(1,x)=1$), and denote the cumulant generating function $g(t,x)=\log(G(t,x))$. We will continue to assume that x is a scalar following $x_t=\mu_x+\phi_x\widetilde{\Delta c}_t$. The relevant equations for the value function and log-SDF are modified as follows:

$$\begin{aligned}\exp((1-\gamma)\psi_t) &= E_t\left[\exp\left((1-\gamma)\left(v_{t+1}+\Delta c_{t+1}+\frac{1}{1-\gamma}g(1-\gamma,x_{t+1})\right)\right)\right]\\ m_{t+1} &= \log(\beta)-\rho\Delta c_{t+1}+(\rho-\gamma)(v_{t+1}-\psi_t+\Delta c_{t+1})+g(-\gamma,x_{t+1})\end{aligned}$$

and their steady state values, given that $\bar{x}=\mu_x$, are

$$\begin{aligned}\bar{v} &= \frac{1}{1-\rho}\log\left(\frac{1-\beta}{1-\beta \mathrm{e}^{(1-\rho)\left(\overline{\Delta c}+\frac{1}{1-\gamma}g(1-\gamma,\bar{x})\right)}}\right)\\ \bar{\psi} &= \bar{v}+\overline{\Delta c}+\frac{1}{1-\gamma}g(1-\gamma,\bar{x})\\ \bar{m} &= \log(\beta)-\rho\overline{\Delta c}+\frac{\gamma-\rho}{1-\gamma}g(1-\gamma,\bar{x})+g(-\gamma,\bar{x})\end{aligned}$$

To solve for dynamics, linearize g wrt. x at $t=-\gamma$ and $t=1-\gamma$:

$$\begin{aligned}g(-\gamma,x) &\approx g(-\gamma,\bar{x})+\theta_{(-\gamma)}\tilde{x}\\ g(1-\gamma,x) &\approx g(1-\gamma,\bar{x})+\theta_{(1-\gamma)}\tilde{x}\end{aligned}$$

where $\theta_{(t)}=\frac{\partial g(t,\bar{x})}{\partial x}$. Linearized equations then become

$$\begin{aligned}\tilde{v}_t &= \kappa\tilde{\psi}_t\\ \tilde{\psi}_t &= E_t\left[\tilde{v}_{t+1}+\widetilde{\Delta c}_{t+1}+(1/(1-\gamma))\theta_{(1-\gamma)}\tilde{x}_{t+1}\right]\\ \tilde{m}_{t+1} &= -\gamma\widetilde{\Delta c}_{t+1}+(\rho-\gamma)(\tilde{v}_{t+1}-\tilde{\psi}_t)+\theta_{(-\gamma)}\tilde{x}_{t+1}\end{aligned}$$

where $\kappa=\beta\exp\left((1-\rho)\left(\overline{\Delta c}+\frac{1}{1-\gamma}g(1-\gamma,\mu_x)\right)\right)$. Everything else is the same as in the previous case, and following the same argument we can derive effective inverse IES:

$$\hat{\rho}=\rho+\frac{\gamma-\rho}{\gamma-1}\theta_{(1-\gamma)}\phi_x-\theta_{(-\gamma)}\phi_x$$

and then η_{ck} is (the positive) solution to

$$\hat{\rho}\lambda_2\eta_{ck}^2 + (\hat{\rho} - \lambda_2\lambda_3 - \hat{\rho}\lambda_1)\,\eta_{ck} + \lambda_1\lambda_3 = 0$$

Using the method of undetermined coefficients, η_{vk} can be derived to be

$$\eta_{vk} = \frac{\kappa\left(1 + \frac{1}{1-\gamma}\theta_{(1-\gamma)}\phi_x\right)\eta_{ck}\left(\lambda_1 - \lambda_2\eta_{ck} - 1\right)}{1 - \kappa\left(\lambda_1 - \lambda_2\eta_{ck}\right)}$$

Then one can show that the innovation to log-SDF is

$$m_{t+1} - E_t[m_{t+1}] = \Big(\gamma(1-\eta_{ck}) + (\gamma-\rho)(-\eta_{vk}) + \theta_{(-\gamma)}(-\phi_x)(1-\eta_{ck})\Big)\left(-\epsilon_{t+1}\right)$$

which can again be used to decompose the risk premium, with the share of long run risk attributable to news about x being $\frac{\left(\frac{1}{1-\gamma}\theta_{(1-\gamma)}\phi_x\right)}{\left(1+\frac{1}{1-\gamma}\theta_{(1-\gamma)}\phi_x\right)}$.

A.5 Data sources

Data moments in table 2 for macroeconomic variables are obtained from quarterly national accounts data constructed by the U.S. Bureau of Economic Analysis and published in the St. Louis Fed FRED database. The sample period is 1947Q1 - 2016Q2. Output and investment growth (Δy, Δi) are computed as logarithmic growth rates of GDP and gross private domestic fixed investment quantity indices (NIPA table 1.1.3) divided by population (NIPA table 7.1). Consumption growth (Δc) is computed as a weighted average of logarithmic growth rates in quantity indices for nondurables and services consumption (NIPA table 1.1.3) divided by population, with weights determined by nominal shares of both consumption components in combined nominal nondurable+services consumption (NIPA table 1.1.5), i.e. using the Tornqvist index method (however, simply summing both series in real chained dollars yields almost identical results).

Data for financial returns are constructed from monthly dataset on Fama-French 3 factors published on Kenneth French's website[12]. In place of the return on capital/firm stock (R^s) I use the market return (i.e. the return on value-weighted portfolio of all firms listed at NYSE, AMEX or NASDAQ), while the risk-free rate (R^b) is represented by the return on 1-month Treasury bill. Returns are expressed in real terms by subtracting CPI inflation (series CPIAUCSL from FRED) and aggregated to quarterly frequency by summing monthly returns over the given quarter. The resulting sample period is 1947Q1 - 2016Q3.

[12] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

References

Adjemian, Stephane, Houtan Bastani, Frederic Karame, Michel Juillard, Junior Maih, Ferhat Mihoubi, George Perendia, Johannes Pfeifer, Marco Ratto, and Sebastien Villemot. 2011, April. "Dynare: Reference Manual Version 4." Dynare working papers 1, CEPREMAP.

Albuquerque, Rui, Martin Eichenbaum, Victor Xi Luo, and Sergio Rebelo. 2016. "Valuation Risk and Asset Pricing." *The Journal of Finance* 71 (6): 2861–2904 (December).

Andreasen, Martin M., Jesus Fernandez-Villaverde, and Juan Rubio-Ramirez. 2013, April. "The Pruned State-Space System for Non-Linear DSGE Models: Theory and Empirical Applications." Working Paper 18983, National Bureau of Economic Research.

Balduzzi, Pierluigi, and Tong Yao. 2007. "Testing heterogeneous-agent models: an alternative aggregation approach." *Journal of Monetary Economics* 54 (2): 369–412 (March).

Bansal, Ravi, and Amir Yaron. 2004. "Risks for the long run: A potential resolution of asset pricing puzzles." *The Journal of Finance* 59 (4): 1481–1509.

Braun, R. Anton, and Tomoyuki Nakajima. 2012. "Uninsured Countercyclical Risk: An Aggregation Result and Application to Optimal Monetary Policy." *Journal of the European Economic Association* 10 (6): 1450–1474 (December).

Brav, Alon, George M. Constantinides, and Christopher C. Geczy. 2002. "Asset Pricing with Heterogeneous Consumers and Limited Participation: Empirical Evidence." *Journal of Political Economy* 110 (4): 793–824.

Campanale, Claudio, Rui Castro, and Gian Luca Clementi. 2010. "Asset pricing in a production economy with Chew-Dekel preferences." *Review of Economic Dynamics* 13 (2): 379–402 (April).

Campbell, John Y. 1994. "Inspecting the mechanism: An analytical approach to the stochastic growth model." *Journal of Monetary Economics* 33 (3): 463–506 (June).

Carceles-Poveda, Eva. 2009. "Asset prices and business cycles under market incompleteness." *Review of Economic Dynamics* 12 (3): 405–422.

Carceles-Poveda, Eva, and Daniele Coen-Pirani. 2009. "Shareholders' Unanimity with Incomplete Markets*." *International Economic Review* 50 (2): 577–606 (May).

Cochrane, John H. 2008. "Financial Markets and the Real Economy." In *Handbook* *of the Equity Risk Premium*, edited by Rajnish Mehra, Handbooks in Finance, 237 – 325. San Diego: Elsevier.

Cogley, Timothy. 2002. "Idiosyncratic risk and the equity premium: evidence from the consumer expenditure survey." *Journal of Monetary Economics* 49 (2): 309 – 334.

Constantinides, George M., and Anisha Ghosh. 2017. "Asset Pricing with Countercyclical Household Consumption Risk." *The Journal of Finance* 72 (1): 415–460 (February).

Constantinides, George M, and Darrell Duffie. 1996. "Asset pricing with heterogeneous consumers." *Journal of Political economy*, pp. 219–240.

Croce, Mariano M. 2014. "Long-run productivity risk: A new hope for production-based asset pricing?" *Journal of Monetary Economics* 66 (September): 13–31.

Epstein, Larry G., and Stanley E. Zin. 1989. "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework." *Econometrica* 57 (4): pp. 937–969.

Gomes, Francisco, and Alexander Michaelides. 2008. "Asset Pricing with Limited Risk Sharing and Heterogeneous Agents." *Review of Financial Studies* 21 (1): 415–448.

Guvenen, Fatih, Serdar Ozkan, and Jae Song. 2014. "The Nature of Countercyclical Income Risk." *Journal of Political Economy* 122 (3): 621–660 (June).

Hansen, Lars Peter, and Kenneth J. Singleton. 1982. "Generalized instrumental variables estimation of nonlinear rational expectations models." *Econometrica* 50 (5): 1269–1286.

Havranek, Tomas. 2015. "Measuring Intertemporal Substitution: The Importance of Method Choices and Selective Reporting." *Journal of the European Economic Association* 13 (6): 1180–1204 (December).

Heaton, John, and Deborah J Lucas. 1996. "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing." *Journal of Political Economy*, pp. 443–487.

Herskovic, Bernard, Bryan Kelly, Hanno Lustig, and Stijn Van Nieuwerburgh. 2016. "The common factor in idiosyncratic volatility: Quantitative asset pricing implications." *Journal of Financial Economics* 119 (2): 249–283 (February).

Kaltenbrunner, Georg, and Lars A. Lochstoer. 2010. "Long-Run Risk through Consumption Smoothing." *Review of Financial Studies* 23 (8): 3190–3224.

Kogan, Leonid, and Dimitris Papanikolaou. 2012. "Economic Activity of Firms and Asset Prices." *Annual Review of Financial Economics* 4 (1): 361–384.

Krebs, Tom, and Bonnie Wilson. 2004. "Asset returns in an endogenous growth model with incomplete markets." *Journal of Economic Dynamics and Control* 28 (4): 817–839 (January).

Kreps, David M., and Evan L. Porteus. 1978. "Temporal Resolution of Uncertainty and Dynamic Choice Theory." *Econometrica* 46 (1): 185 (January).

Krueger, Dirk, and Hanno Lustig. 2010. "When is Market Incompleteness Irrelevant for the Price of Aggregate Risk (and when is it not)?" *Journal of Economic Theory* 145 (1): 1–41.

Krusell, Per, and Anthony Smith. 1997. "Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns." *Macroeconomic dynamics* 1:387–422.

Krusell, Per, and Anthony A. Smith. 1998. "Income and Wealth Heterogeneity in the Macroeconomy." *Journal of Political Economy* 106 (5): 867.

Ludvigson, Sydney C. 2013. "Advances in Consumption-Based Asset Pricing: Empirical Tests." In *Handbook of the Economics of Finance*, Volume 2, 799–906.

Mankiw, N Gregory. 1986. "The equity premium and the concentration of aggregate shocks." *Journal of Financial Economics* 17 (1): 211–219.

McKay, Alisdair. 2017. "Time-Varying Idiosyncratic Risk and Aggregate Consumption Dynamics." *Journal of Monetary Economics* 88:1–14.

Mehra, Rajnish, and Edward C Prescott. 1985. "The equity premium: A puzzle." *Journal of Monetary Economics* 15 (2): 145–161.

Pijoan-Mas, Josep. 2007. "Pricing Risk in Economies with Heterogeneous Agents and Incomplete Markets." *Journal of the European Economic Association* 5 (5): 987–1015 (September).

Pohl, Walter, Karl Schmedders, and Ole Wilms. 2018. "Higher-Order Effects in Asset Pricing Models with Long-Run Risks." *The Journal of Finance*. forthcoming.

Restoy, Fernando, and G. Michael Rockinger. 1994. "On Stock Market Returns and Returns on Investment." *The Journal of Finance* 49 (2): 543–556.

Rudebusch, Glenn D., and Eric T. Swanson. 2012. "The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks." *American Economic Journal: Macroeconomics* 4 (1): 105–43.

Schmidt, Lawrence D. W. 2014. "Climbing and Falling Off the Ladder: Asset Pricing Implications of Labor Market Event Risk." August.

Storesletten, Kjetil, Chris I. Telmer, and Amir Yaron. 2004. "Cyclical Dynamics in Idiosyncratic Labor Market Risk." *Journal of Political Economy* 112 (3): 695–717 (June).

Storesletten, Kjetil, Christopher I Telmer, and Amir Yaron. 2007. "Asset pricing with idiosyncratic risk and overlapping generations." *Review of Economic Dynamics* 10 (4): 519–548.

Takahashi, Yuta, Lawrence Schmidt, Konstantin Milbradt, Ian Dew-Becker, and David Berger. 2016. "Layoff risk, the welfare cost of business cycles, and monetary policy." Technical Report 1293, Society for Economic Dynamics.

Tallarini, Thomas D. 2000. "Risk-sensitive real business cycles." *Journal of Monetary Economics* 45 (3): 507 – 532.

Telmer, Chris I. 1993. "Asset-pricing Puzzles and Incomplete Markets." *The Journal of Finance* 48 (5): 1803–1832.

Toda, Alexis Akira. 2014. "Incomplete market dynamics and cross-sectional distributions." *Journal of Economic Theory* 154:310 – 348.

van Binsbergen, Jules H., Jesus Fernandez-Villaverde, Ralph S.J. Koijen, and Juan Rubio-Ramirez. 2012. "The term structure of interest rates in a DSGE model with recursive preferences." *Journal of Monetary Economics* 59 (7): 634–648 (Nov).

Weil, Philippe. 1989. "The equity premium puzzle and the risk-free rate puzzle." *Journal of Monetary Economics* 24 (3): 401–421.

Werning, Ivan. 2015, August. "Incomplete Markets and Aggregate Demand." Working Paper 21448, National Bureau of Economic Research.

Abstrakt

Tento článek zkoumá rizikovou prémii v ekonomice s nekompletními trhy a domácnostmi čelícími idiosynkratickému riziku ve spotřebě. Pokud je rozptyl idiosynkratického rizika proměnlivý v průběhu hospodářského cyklu a domácnosti preferují dřívější rozřešení nejistoty, pak ceny finančních aktiv budou ovlivněny nejen zprávami o současné a očekávané budoucí spotřebě (jako je tomu v modelech s reprezentativní domácností), ale také zprávami o současných a budoucích změnách distribuce individuální spotřeby napříč domácnostmi. V článku zkoumám, jestli tento dodatečný efekt může pomoci vysvětlit vysokou rizikovou prémii v produkční ekonomice, ve které je proces pro agregátní spotřebu endogenní a potenciálně může být ovlivněn přítomností idiosynkratického rizika. Analýzou neoklasického růstového modelu kombinovaného s Epstein-Zin preferencemi a jednoduše řešitelnou formou heterogenity domácností jsem zjistil, že proticyklické idiosynkratické riziko zvyšuje rizikovou prémii, ale zároveň snižuje efektivní ochotu domácností k intertemporální substituci, čímž se změní dynamika agregátní spotřeby. Pokud umožníme vyšší elasticitu intertemporální substituce na individuální úrovni, pak je díky flexibilitě Epstein-Zin preferencí možné zvýšit rizikovou prémii beze změny dynamiky agregátních veličin.

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