# Sovereign Default: The Role of Expectations*

João Ayres
University of Minnesota

Gaston Navarro
Federal Reserve Board

Juan Pablo Nicolini
Federal Reserve Bank of Minneapolis and Universidad Di Tella

Pedro Teles
Banco de Portugal, Catolica Lisbon School of Business & Economics, CEPR

January 2016

**Abstract**

In the standard model of sovereign default, as in Aguiar and Gopinath (2006) or Arellano (2008), default is driven by fundamentals alone. There is no independent role for expectations. We show that a small variation of that model is consistent with multiple interest rate equilibria. Some of those equilibria correspond to the ones identified by Calvo (1988), where default is likely because rates are high, and rates are high because default is likely. The model is used to discuss lending policies similar to the ones announced by the European Central Bank in 2012.

*We thank Patrick Kehoe for many useful conversations. We also thank Fernando Alvarez, Manuel Amador, Cristina Arellano, V.V. Chari, In-Koo Cho, Jonathan Eaton, Veronica Guerrieri, Jonathan Heathcote, Andy Neumeyer, Fabrizio Perri, Martin Uribe, and Vivian Yue for very useful comments. Nicolini and Teles would like to acknowledge the support of FCT as well as the ADEMU project, "A Dynamic Economic and Monetary Union", funded by the European Union's Horizon 2020 Program under grant agreement N° 49396. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis, the Federal Reserve System, Banco de Portugal or the European System of Central Banks. E-mail adresses: ayre0041@umn.edu, gaston.m.navarro@frb.gov, juanpa@minneapolisfed.org, pteles@ucp.pt.

*Keywords:* Sovereign default; interest rate spreads; multiple equilibria.

*JEL Codes:* E44, F34.

# 1 Introduction

Are sovereign debt crises caused by bad fundamentals alone, or do expectations play an independent role? The main point of the paper is that both fundamentals and expectations can play important roles. High interest rates can be triggered by self-confirming expectations. In particular, high interest rates can induce high default probabilities that in turn justify the high rates. It is also the case, however, that the self fulfilling high rate equilibria arise only when debt levels are relatively high. The model analyzed can help to explain the large and abrupt increases in spreads during sovereign debt crises, particularly in countries that have accumulated large debt levels, as seen in the recent European experience. It can also justify the policy response by the European Central Bank, to be credited for the equally large and abrupt reduction in sovereign spreads.[1]

The literature on sovereign debt crises is ambiguous on the role of expectations. In a model with rollover risk, Cole and Kehoe (2000) have established that sunspots can play a role that is strengthened by bad fundamentals. Using a different mechanism, Calvo (1988) also shows that there are multiple -low and high- interest rate equilibria. The reason is that, although interest rates may be high because of high default probabilities, it is also the case that high interest rates induce high default probabilities. This gives rise to equilibria with high rates/likely default and low rates/unlikely default. In contrast with the results in those models, in the standard quantitative model of sovereign default, as in Aguiar and Gopinath (2006) or Arellano (2008), there is a single low interest rate equilibrium.

[1] At the ECB press conference of September 6 of 2012, where the Outright Monetary Transactions program was announced, President Draghi explicitly stated his beliefs of a self-fulling nature behind the increase in spreads as justification for the program. In his words: "[...] the assessment of the Governing Council is that we are in a situation now where you have large parts of the euro area in what we call a “bad equilibrium”, namely an equilibrium where you may have self-fulfilling expectations that feed upon themselves and generate very adverse scenarios. So, there is a case for intervening, in a sense, to “break” these expectations[...]". See the announcement here: http://www.ecb.europa.eu/press/pressconf/2012/html/is120906.en.html

In this paper, we take the model of Aguiar and Gopinath (2006) and Arellano (2008), which builds on the model of Eaton and Gersovitz (1981), and make minor changes in the modeling choices concerning the timing of moves by debtors and creditors and the actions that they may take. In so doing, we are able to produce both high and low rate equilibria. The reason for the multiplicity is the one identified by Calvo (1988) and more recently analyzed in Lorenzoni and Werning (2013).

The change in the modeling choices is minor because it is not clear how direct evidence can be used to discriminate across them. The timing and action assumptions concern the sequence of moves by creditors and borrower and whether the borrower chooses current debt or debt at maturity. The actual institutional details behind bond auctions do not seem to provide direct evidence on those assumptions. Bond auctions are for announced quantities of discount bonds, but those quantities are many times revised. So it is not clear whether the choice at those auctions is for current debt or debt at maturity. Even if quantities of discount bonds were not revised, there are multiple auctions in a reference period, and how many auctions there are is a choice variable. In some auctions, lenders place price-quantity schedules, but those quantities are the intended purchases of the lender, not the aggregate quantities. The schedules in the models in which the borrower moves first have the interest rate be a function of the aggregate quantity, and those are not directly observed anywhere. On the other hand, ample indirect evidence on the modelling assumptions is provided by large and abrupt movements in spreads, apparently unrelated to fundamentals, during sovereign debt crises.

Our theoretical exploration of self-fulfilling equilibria in interest rate spreads is motivated by two particular episodes of sovereign debt crises. The first is the Argentine crisis of 1998-2002. Back in 1993, Argentina had regained access to international capital markets. Argentina's debt to GDP ratio was roughly between 35% and 45% during the period—very low by international standards. The average yearly growth rate of GDP was around 5%. But the average country risk spread on dollar denominated bonds for the period 1993-1999, relative to the US bond, was 7%. Notice that a 7% spread on a 35% debt to GDP ratio amounts to almost 2.5% of GDP on extra interest payments per year.[2] Accumulated over the 1993-1999 period, this is 15% of GDP, almost half the debt to GDP ratio of Argentina in 1993. An obvious question arises: if Argentina

[2]This calculation unrealistically assumes one-period maturity bonds only. Its purpose is only to illustrate the point in a simple way.

had faced lower interest rates, would it have defaulted in 2002?

The second episode is the recent European sovereign debt crisis that started in 2010 and has receded substantially since the policy announcements by the European Central Bank (ECB) in September 2012. The spreads on Italian and Spanish public debt, very close to zero since the introduction of the euro and until April 2009, were higher than 5% by the summer of 2012, when the ECB announced the program of Outright Monetary Transactions (OMTs). The spreads were considerably higher in Portugal, and especially in Ireland and Greece. With the announcement of the OMTs, according to which the central bank stands ready to purchase euro area sovereign debt in secondary markets, the spreads in most of those countries slid down to less than 2%, even though the ECB did not actually intervene. The potential self-fulfilling nature of the events leading to the high spreads of the summer 2012 was explicitly used by the president of the ECB to justify the policy.[3]

The model is of a two-period small open economy with a random endowment. A representative agent can borrow noncontingent bonds and cannot commit to repayment. Defaulting carries a penalty. The borrower faces atomistic risk-neutral foreign creditors, so that their expected return, taking default probabilities into account, has to be equal to the risk-free interest rate. The timing and action assumptions are the following. In the beginning of the period, given the level of debt gross of interest and the realization of the endowment, the borrower decides whether to default. If there is default, the endowment is low. Otherwise, creditors move first and offer their limited funds at some interest rate. The borrower moves next and borrows from the low-rate creditors up to some total optimal debt level. In equilibrium, the creditors all charge the same rate, which is the one associated with the probability of default for the optimal level of debt chosen by the country. With these timing assumptions, there are multiple interest rate equilibria. High interest rates can generate high default rates, which in turn justify high interest rates. In equilibria such as these, there is a sense in which interest rates are "too high."

With this timing, when deciding how much to borrow, the borrower takes the interest rate as given. This does not necessarily mean that the borrower behaves like a small agent. In a version of the model with more than two periods, even if the borrower would take current prices as given, the effects of current choices on future prices would

[3] The decision has raised controversy. In 2014, the German Constitutional Court ruled the OMT to be incompatible with the constitution.

still be taken into account. The borrower in this model is just not benefiting from a first mover advantage. A similar timing assumption in Bassetto (2005) also generates multiple Laffer curve equilibria. In Bassetto, if the government were to move first and pick the tax, there would be a single low tax equilibrium. Instead, if households move first choosing how much labor to supply, there is also a high tax equilibrium. Bassetto argues that the assumption that the government is a large agent is unrelated to the timing of the moves.

With this timing in which creditors move first, and also when moves are simultaneous, there are multiple equilibria, regardless of whether the borrower chooses current debt or debt at maturity. In contrast, under the standard timing in which the borrower moves first, this is a key distinction .

The timing assumption in Aguiar and Gopinath (2006) and Arellano (2008) is that the borrower moves first, before the creditors. They also assume that the borrower chooses the debt level at maturity, including interest payments. Creditors move next and respond with a schedule that specifies a single interest rate for each level of debt gross of interest. By moving first and choosing the debt at maturity, gross of interest, the borrower is able to select a point in the schedule. The borrower will optimally pin down the low interest rate/low probability of default. It follows that there is a single equilibrium. The first mover advantage allows the borrower to coordinate the creditor's actions on the low interest rate equilibrium.

There is no first mover advantage if instead, the borrower, in spite of moving first, chooses current debt, rather than debt at maturity. In this case, interest rate schedule will be a function of current debt, and there will be multiple schedules.[4] The reason is that, given current debt, if the interest rate is high, so is debt at maturity, and therefore the probability of default is also high. This is the spirit of the analysis in Calvo (1988), as well as Lorenzoni and Werning (2013).

Lorenzoni and Werning (2013) analyze a dynamic version of Calvo's model with exogenous public deficits, and argue against the possibility of the government choosing debt at maturity. For that, they build a one period game with an infinite number of subperiods and assume that the government cannot commit not to reissue debt in those subperiods. As a result, the government is unable to select a point on the interest rate

[4] In Eaton and Gersovitz (1981), even if that is the assumption on the actions of the country, they dismiss the multiplicity by assumption (discussed further in Section 2.3).

schedule, behaving as a price taker.[5] The result in the game of Lorenzoni and Werning could thus be interpreted as a flipping of the timing of moves of borrower and creditors. The possibility to always reissue would be as if the borrower was moving second, after the creditors, as in our timing.

As mentioned above, the reason for expectation-driven high interest rate equilibria in these models is different from the one in Cole and Kehoe (2000). Still, in both setups it is the timing of moves that is crucial to generate multiplicity. In Cole and Kehoe, there is multiplicity when the choice of how much debt to issue takes place before the decision to default. In that case, it may be individually optimal for the creditors not to roll over the debt, which amounts to charging arbitrarily high interest rates. This may induce default, confirming the high interest rates. In our model, there is no rollover risk because the decision of default is at the beginning of the period. Still, a timing assumption similar to the one in Cole and Kehoe generates the multiplicity. As creditors move first, it can be individually optimal to ask for high rates. That will induce a high probability of default, confirming the high rates.

For standard distributions of the endowment, the high rate equilibria have properties that make them vulnerable to reasonable refinements, which we provide in an appendix. Those high rates can be in parts of the supply curve in which the rates decrease with the level of debt. If that is the case, then the total gross service of the debt also decreases with an increase in the level of debt. For those high rates, creditors also jointly benefit from lowering interest rates because of their effect on probabilities of default. These are all features of the high rate equilibria in Calvo (1988). But as we show, multiplicity does not disappear even if those equilibria are refined away. To show this, we consider bimodal distributions for the endowment, with good and bad times. With those distributions, there are low and high rate equilibria, equally robust, for the same level of debt. The set of equilibria has the feature that for low levels of debt, there is only one equilibrium. Interest rates are low and increase slowly with the level of debt. As debt becomes relatively high, then there are both low and high rate equilibria. For even higher levels of debt, there is a single high rate equilibrium, until eventually there is none.[6] As we explain in detail in the paper, we consider these bimodal distributions

[5] The result is analogous to the one in the durable good monopoly literature, that without intertemporal costs, a monopoly, competing with its own future self, behaves as a price taker.

[6] Lorenzoni and Werning (2013) also consider such distributions. They do not give them empirical content, as we do.

as reflecting the likelihood of relatively long periods of stagnation, as currently discussed in Europe, in a way that resembles the Markov-switching processes for output popularized by Hamilton (1989).[7] We emphasize the role of large debt levels and the plausibility of long periods of stagnation as drivers of the multiplicity.

In the region where the interest rates are unnecessarily high, policy can be effective in selecting a low rate equilibrium. A large lender can accomplish the missing coordination by lending up to a maximum amount at a penalty rate. In equilibrium, only private creditors would be lending. This may help us understand the role of policies such as the OMTs introduced by the ECB, following the announcement by its president that it would do "whatever it takes" to avoid a sovereign debt crisis in the euro area. The analysis highlights a key role for quantity restrictions in the design of policies aimed at eliminating the "bad" equilibria, suggesting that to do "whatever it takes," understood as no limit on bond purchases, does not follow from the model.

In this paper, we only analyze a simple two-period model to highlight the importance of both timing and action assumptions for multiple interest rate equilibria to arise, clarifying apparent inconsistencies in the literature. By exposing the importance of these assumptions, we argue for the empirical relevance of that multiplicity. In a companion paper (Ayres, Navarro, Nicolini and Teles, 2015), a quantitative exercise is performed in a calibrated dynamic model in which a sunspot variable is introduced, triggering coordination on high or low interest rates. The model is shown to be consistent with a sovereign debt crisis unraveling, in particular when debt is relatively large and the probability of a relatively long period of stagnation is high. The fact that debt choices are optimal and the model is fully dynamic allows for the discussion of the role of the endogenous decision to borrow on the likelihood and characteristics of the debt crisis.

Finally, we should mention that there is a large literature extending Calvo (1988) and Cole and Kehoe (2000) in directions other than the ones we are concerned in this paper. See for example Aguiar, Amador, Farhi and Gopinath (2014), Bocola and Dovis (2015), Conessa and Kehoe (2012), Corsetti and Dedola (2013) and Roch and Uhlig (2015) among others.

[7] See also the evidence in Jones and Olken (2008) for an international perspective.

# 2 The model

The model is a simple two-period model, where analytical results can be derived and some of the features of the model can be seen clearly. In particular, it is easier to understand what drives the multiplicity of spreads and default probabilities that resembles the result in Calvo (1988).

We analyze a two-period endowment economy populated by a representative agent that draws utility from consumption in each period, and by a continuum of risk-neutral foreign creditors. Each creditor has limited capacity, but there are enough of them so that there is no constraint on the aggregate credit capacity. The period utility function of the representative agent, $U$, is assumed to be strictly increasing and strictly concave and to satisfy standard Inada conditions. The endowment is assumed to be equal to 1 in the first period. That is the lower bound of the support of the distribution of the endowment in the second period. Indeed, uncertainty regarding future outcomes is described by a stochastic endowment $y \in [1, Y]$, with density $f(y)$ and corresponding cdf $F(y)$. The outstanding initial level of debt is assumed to be zero, and in period one, the representative agent can borrow $b$ in a noncontingent bond in international financial markets. The risk-neutral gross international interest rate is $R^*$. In period two, after observing the realization of the shock, the borrower decides to either pay the debt gross of interest, $Rb$, or default. If there is default, consumption is equal to the lower bound of the endowment process, 1. Note that there may be contingencies under which the borrower chooses to default, and the interest rate charged by foreign creditors, $R$, may differ from the risk-free rate $R^*$.

The timing of moves is as follows. In the first period, each creditor $i \in [0, 1]$ offers the limited funds at the gross interest rate $R_i$. The borrower moves next and picks the level of debt $b = \int_0^1 b_i di$, where $b_i$ is how much is borrowed from each creditor. The borrower's best response is to borrow from the low interest rate lenders first. In order for lenders to make zero profits in equilibrium, the interest rates they charge will have to be the same, $R_i = R$. We focus on symmetric outcomes where if $R_i = R_j$, then $b_i = b_j$. Then, $b_i = b$ for all $i \in [0, 1]$, so $\int_0^1 b_i R_i di = a = Rb$.

In the second period, the borrower decides whether to default or pay the debt in full. The borrower decides to default if and only if $U\left(y-\int_0^1 b_i R_i di\right) \leq U(1)$, or

$$y \leq 1+\int_0^1 b_i R_i di.$$

Accordingly, default happens whenever

$$y \leq 1+bR,$$

which defines a default threshold for output. The probability of default is then $F[1+bR]$.

Since creditors are risk neutral, the expected return of lending to the borrower in this economy must be the same as $R^*$, so

$$R^* = R[1-F(1+bR)]. \tag{1}$$

This defines a locus of points $(b, R)$ such that each point solves the problem of the creditors, which can be interpreted as a supply curve of funds. The mapping from debt levels to interest rates is a correspondence because, in general, for each $b$ there are multiple $R$s that satisfy equation (1). Multiple functions can be selected with the points of the correspondence. We call those functions interest rate schedules.

The optimal choice of debt by the borrower is the one that maximizes utility:

$$U(1+b)+\beta\left[F(1+bR)U(1)+\int_{1+bR}^{Y} U(y-bR)f(y)dy\right], \tag{2}$$

subject also to an upper bound restriction on the maximum level of debt. Absent this condition, the optimal choice would be to borrow an arbitrarily large amount and default with probability one. The supply of debt would be zero in equilibrium.

The marginal condition, for an interior solution, is

$$U'(1+b) = R\beta \int_{1+bR}^{Y} U'(y-bR)f(y)\,dy. \tag{3}$$

The optimal choice of debt for a given interest rate defines a locus of points $(b, R)$ that can be interpreted as a demand curve for funds. The possible equilibria will be the points where the demand curve intersects the supply curve above described by (1).

An equilibrium in this economy can then be defined as follows:

**Definition 1** *An equilibrium is an interest rate $\widetilde{R}$ and a debt level $\widetilde{b}$ such that (i) given $\widetilde{R}$, $\widetilde{b}$ maximizes (2); and (ii) the arbitrage condition (1) is satisfied.*

## 2.1 Multiple equilibria

As mentioned above, there are in general multiple equilibria in this model—low rate equilibria and high rate equilibria—that resemble the multiple equilibria in Calvo (1988).

We now analyze the supply curve defined implicitly by (1). For that purpose, it is useful to define the function for the expected return on the debt:

$$h(R;b) = R\left[1 - F(1 + bR)\right],$$

which in equilibrium must be equal to the riskless rate, $R^*$. Notice that for $R = 0$, we have $h(0;b) = 0$. If the distribution of the endowment has a bounded support, for $R$ high enough, if $1 + bR \geq Y$, then $h(R;b) = 0$. For many distributions, the function $h(R;b)$ is concave, so that there are at most two solutions for $R^* = h(R;b)$.

In Figure 1, the curve $h(R;b)$ is depicted against $R$, where $F$ is the cumulative normal.[8] An increase in $b$ shifts the curve $h$ downward so that the solutions for $b$ are closer to each other. The function $h(R;b)$ does not need to be concave everywhere; this will depend on the cumulative distribution $F(1 + bR)$. [9]

Figure 2 plots the solutions for $R$ of equation (1) for each level of debt and also for the normal distribution.

The supply curve of Figure 2 has two monotonic schedules. For lower values of the interest rate, there is a flat schedule that is increasing in $b$ (solid line). There is also a steeper decreasing schedule for higher values of the interest rate (dashed line).

The equilibrium must also be on a demand curve for the borrower, obtained from the solution of the problem defined in (2). Figure 3 depicts the two curves: the supply curve (red) and the demand curve (blue). As can be seen, there are two possible equilibria.

The points on the decreasing schedule have particularly striking properties. For those points on the supply curve, not only does the interest rate go down with the

[8] The black vertical dotted lines are grid lines. We kept them in the plots throughout the paper to make the exposition clearer.

[9] The function $h(R;b)$ is concave in $R$ whenever $2f(1 + bR) \geq -f'(1 + bR)\,bR$.

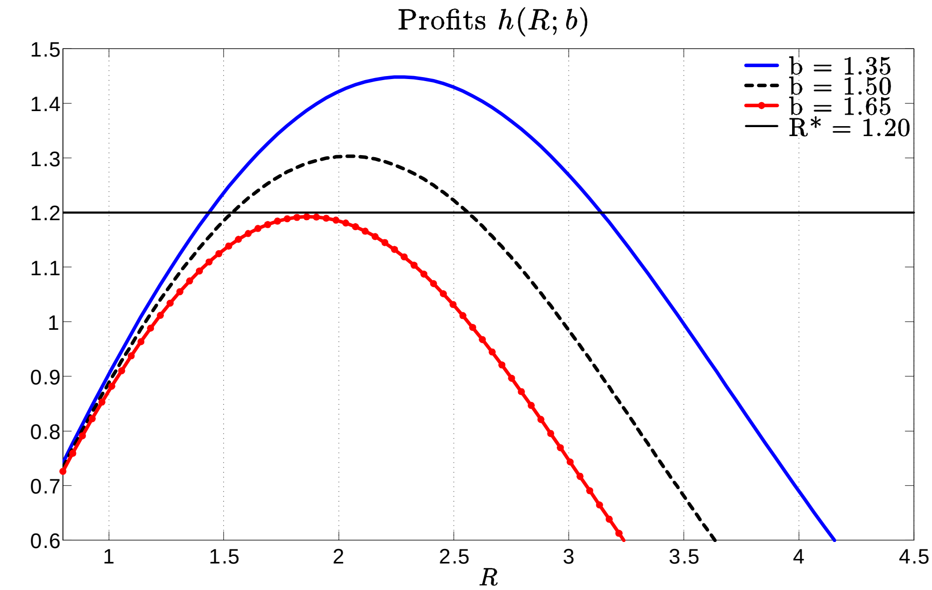



Figure 1: Expected return $h(R;b)$

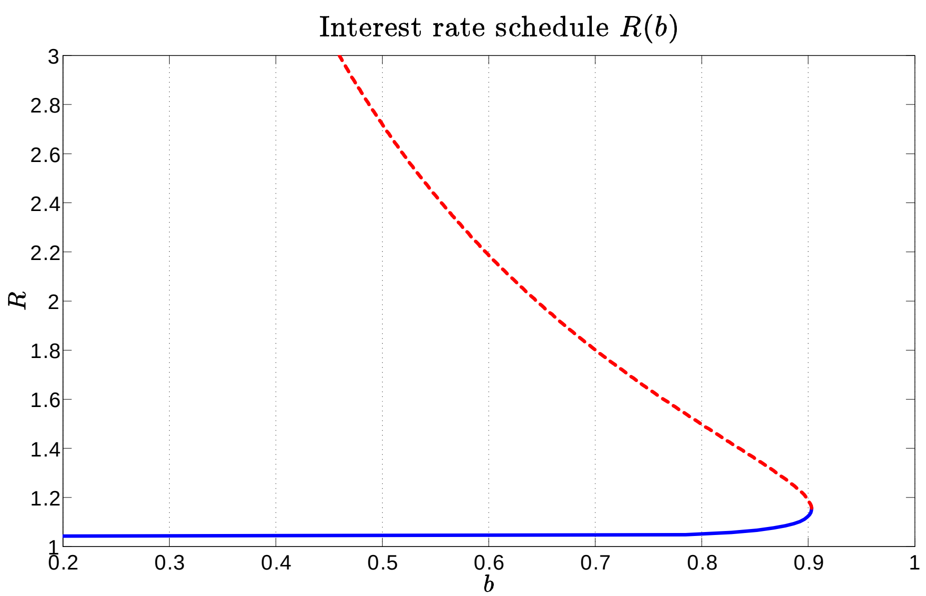



Figure 2: Interest rate schedules

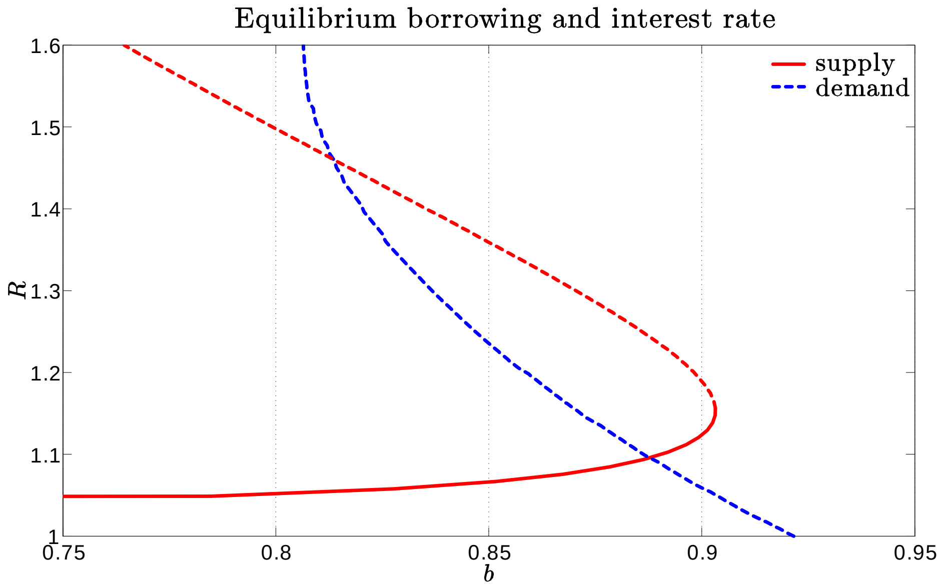



Figure 3: Supply and demand curves

level of debt, $b$, but also the gross service of the debt, $Rb$, decreases with the level of debt, $b$. To see this, notice that from (1), $R$ increases with the level of $Rb$. The points on the decreasing schedule are weak candidates for equilibria in the following sense. Consider a perturbation of a point $(\widehat{R}, \widehat{b})$ in that schedule that consists of the same value for the interest rate but a slightly lower value for debt $(\widehat{R}, \widehat{b} - \varepsilon)$. This point would lie below the schedule. At the point $(\widehat{R}, \widehat{b} - \varepsilon)$, the interest rate is the same as in $(\widehat{R}, \widehat{b})$, but the debt is lower, so the probability of default is also lower. Thus, profits for the creditors are higher than at $(\widehat{R}, \widehat{b})$, where profits are zero. This means that a small reduction in the interest rate is beneficial for both the borrower and the lender, which suggests that these equilibria may not survive reasonable refinements.

In Appendix 1 we perturb the extensive form game by adding an additional stage to the first period. In that second stage of the first period, the borrower can make an offer to a coalition of creditors of a lower, but close, interest rate. Under certain detailed assumptions on the structure of the game, the equilibria on the decreasing schedule can be ruled out.[10] We think of this perturbation as a way of refining the set

[10] There are two important assumptions, as we explain in detail in Appendix 1. First, there must exist a minimal degree of coordination, which, for some equilibria in the decreasing schedule, may be large. Second, the first-period auction must be anonymous, in the sense that ex-ante differences that arise because of the perturbation cannot be observed by the borrower.

of equilibria. It is a concept of refinement by completing the model introducing further details. As with every refinement, there are fragilities with the one we provide. We do not claim that the refinement is the most natural. Rather, we argue that, even if the multiplicity in Calvo (1988) can be refined away, there is still multiplicity that is robust. This is the content of the next section.

#### 2.1.1 A distribution with good and bad times

Equation (1) may have more than two solutions for $R$, for a given $b$, depending on the distribution of the endowment process. One case in which there can be multiple increasing schedules is when the distribution combines two normal distributions—a distribution for good times and a distribution for bad times.

Consider two independent random variables, $y^1$ and $y^2$, both normal with different means, $\mu^1$ and $\mu^2$, respectively, and the same standard deviation, $\sigma$. Now, let the endowment in the second period, $y$, be equal to $y^1$ with probability $p$ and equal to $y^2$ with probability $1 - p$.[11]

If the two means, $\mu^1$ and $\mu^2$, are sufficiently apart, then (1) has four solutions for some values of the debt, as Figure 4 shows. The correspondence between levels of debt and $R$, as solutions to the arbitrage equation above, is plotted in Figure 4, in which $p = 0.8$, $\mu^1 = 4$, $\mu^2 = 6$, and $\sigma = 0.1$. The relatively high probability and the average severity of a disaster can be thought of as a relatively frequent, long period of stagnation. This is in line with the estimation of Hamilton (1989) of high and low growth regime switching processes. For the fully quantitative exercise, see Ayres et al (2015).

Clearly, there are low enough debt levels for which there are only two solutions, so there is only one increasing schedule. But for intermediate levels of debt, the equation has four solutions and therefore multiple increasing schedules. This means that, even if one is restricted not to consider equilibria on decreasing schedules, the model may still exhibit multiplicity. Notice that the multiplicity on the increasing schedules arises for relatively high levels of debt.

The supply curve for this case of a bimodal distribution is indicated by the solid red line in Figure 5. The demand is shown by the dotted blue line in the same figure.

[11] In Appendix 2, we analyze the example of a discrete, two-state, distribution, corresponding to $\sigma = 0$. In this case, there are no decreasing schedules, just increasing ones. The example provides very clear intuition on some of the points we make in the paper..

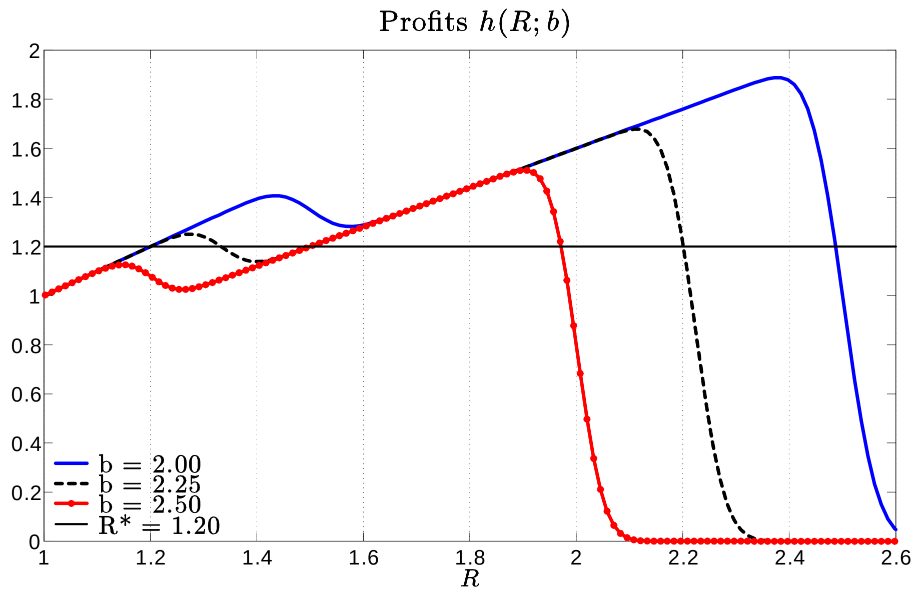



Figure 4: Expected return for the bimodal distribution $h\,(R;b)$

Notice that multiplicity only arises if the demand curve is high enough, so the resulting equilibrium level of debt is high. The demand is discontinuous in this case, since the maximum problem in (2) has two interior local maxima, because of the bimodal distribution. As the interest rate changes, the relative value of utility between the two local maxima changes.

If the debt level is relatively large, multiple equilibria are more likely to arise. This is the case with the bimodal distribution analyzed earlier. It is also the case that, when the value of the debt is close to the maximum and a single mode distribution is perturbed by adding a nonmonotonic function, multiplicity arises. The details are in Appendix 3.

## 2.2 Policy

To illustrate the effects of policy, the case of the bimodal distribution depicted in Figure 5 is considered. The extensions to other cases are straightforward.

Consider that there is a new agent, a foreign creditor that can act as a large lender, with deep pockets.[12] This large lender can offer to lend to the country, at a policy

[12]If the borrower was a small agent rather than a sovereign, any creditor could possibly play this role.

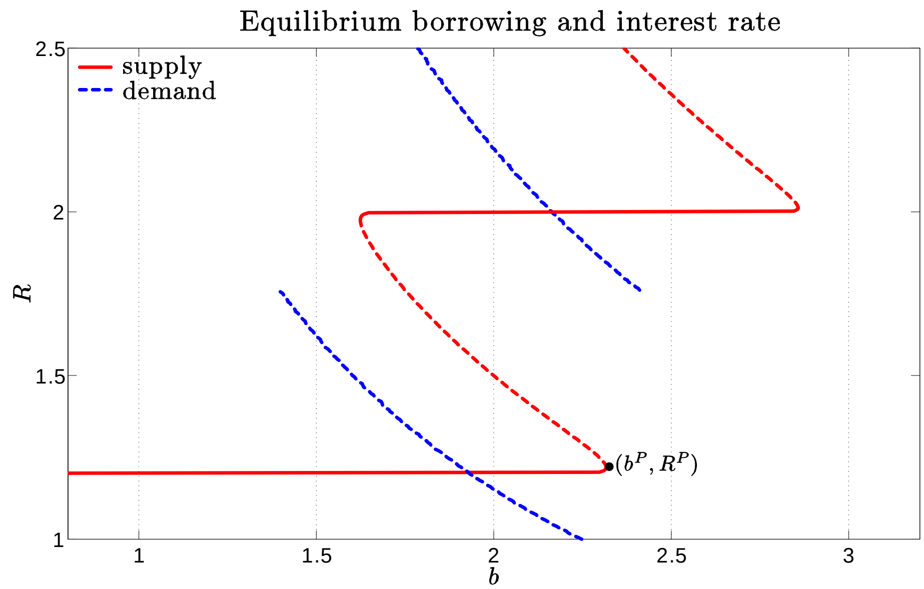



Figure 5: Supply and demand for the bimodal distribution

rate $R^P$ any amount lower than or equal to a maximum level $b^P$. Let $b^P$ and $R^P$ be the debt level and interest rate corresponding to the maximum point of the low (solid line) increasing schedule in Figure 5. In this case, the only equilibrium is the point corresponding to the intersection of demand and supply on the low interest rate increasing schedule. In addition, the amount borrowed from the large lender is zero. The equilibrium interest rate is lower than the one offered by the large lender because at that interest rate $R^P$ and for debt levels strictly below $b^P$, there would be profits.

Notice that the large lender cannot offer to lend any quantity at the penalty rate. Whatever the rate is, the level of lending offered has to be limited by the points on the supply curve. Otherwise, the borrower may borrow a very high amount and then default.

## 2.3 Current debt versus debt at maturity

The borrower in the model analyzed earlier chooses current debt. Would it make a difference if the borrower were to choose debt at maturity, gross of interest? We now consider an alternative game in which the timing of the moves is as before, but now the borrower chooses the value of debt at maturity, which we denote by $a$, rather than the amount borrowed, $b$. Are there still multiple equilibria in this setup? The answer is yes. With this timing of moves, there are multiple interest rate equilibria whether the government chooses the amount borrowed, $b$, or the amount paid back, $a$. This is a relevant question, because in the models of Calvo (1988) and Arellano (2008), the assumption of whether the borrower chooses $b$ or $a$ is key to having uniqueness or multiplicity of equilibria, as will be discussed later.[13]

Here again, the creditors move first and offer the limited funds at gross interest rate $R_i$, $i \in [0, 1]$. The borrower moves next and picks the level of debt at maturity $a = \int_0^1 a_i di$. As before, the rate charged by each creditor will have to be the same in equilibrium. In the second period, the borrower defaults if and only if $y \leq 1 + a$. Arbitrage in international capital markets implies that

$$R^* = R\left[1 - F\left(1 + a\right)\right]. \tag{4}$$

The locus of points $(a, R)$ defined by (4), which we interpret as a supply curve of funds, is monotonically increasing (which is not the case for the supply curve in $b$ and $R$ defined in (1)).

The utility of the borrower is

$$U(1 + \frac{a}{R}) + \beta\left[F(1 + a)U(1) + \int_{1+a}^{Y} U(y - a)f(y)dy\right], \tag{5}$$

where $\frac{1}{R}$ is the price of one unit of $a$ as of the first period. The marginal condition is

$$U'(1 + \frac{a}{R}) = R\beta \int_{1+a}^{Y} U'(y - a)f\left(y\right)dy. \tag{6}$$

The locus of points $(a, R)$ defined by the solution to this maximization problem can be interpreted as a demand curve for funds. Again, this demand curve with the supply curve has multiple intersection points. Provided the choice of $a$ is interior, those points are the solutions to the system of two equations, (4) and (6), but those are the same two equations (1) and (3) that determine the same equilibrium outcomes for $R$ and $b$ for $a = Rb$.

Figure 6 plots the supply curves for $(b, R)$ and $(a, R)$ defined in (1) and (4), respectively, for the normal distribution. It also plots the demand curves defined in (6) and (3) for the logarithmic utility function. With the timing assumed so far, whether the

[13]The key for the different results is the timing assumption, as clarified in Section 3.

Equilibrium outcomes choosing $b$ or $a$

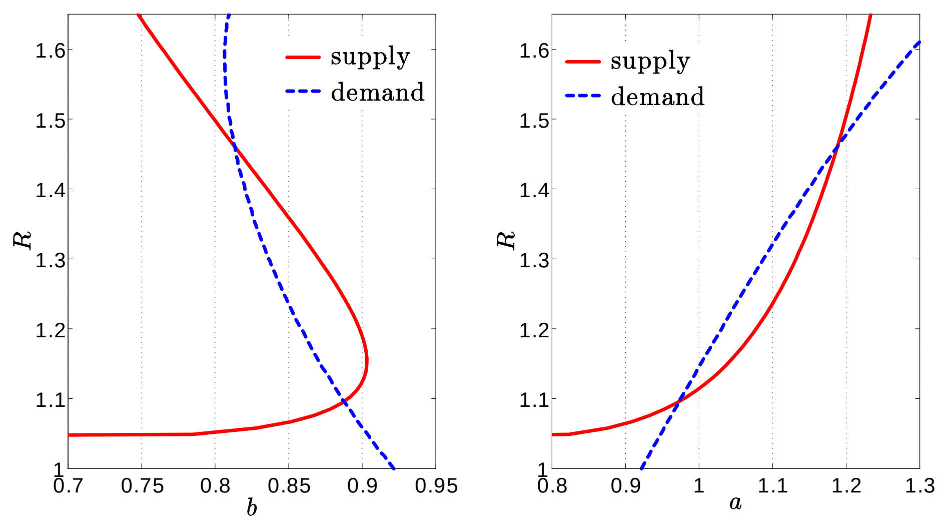



Figure 6: Choosing value of debt at maturity $a$ or amount borrowed $b$

borrower chooses debt net or gross of interest is irrelevant.

# 3 Timing of moves and multiplicity: Related literature

The timing of moves assumed above, with the creditors moving first, amounts to assuming that the borrower in this two-period game takes the current price of debt as given. The more common assumption in the literature is that the borrower moves first, choosing debt levels $b$ or $a$, and facing a schedule of interest rates as a function of those levels of debt, $R = R(b)$ or $R = \frac{1}{q(a)}$, depending on whether the choice is $b$ or $a$, respectively.

Suppose the schedule the borrower faces is $q(a)$, corresponding to the supply curve derived from (4) and depicted in the right-hand panel of Figure 6. This is a monotonically increasing function. Since the borrower can choose $a$, the borrower is always going to choose in the low $R$/low $a$ part of the schedule. The borrower also takes into account the monopoly power in choosing the level of $a$. These are the assumptions in Aguiar and Gopinath (2006) and Arellano (2008). The equilibrium is unique.

Suppose now that the borrower faces the full supply curve as depicted in Figure

2 with an increasing low rate schedule and a decreasing high rate schedule. Then by picking $b$, the borrower is not able to select the equilibrium outcome.[14] There are multiple possible interest rates that make creditors equally happy. The way this can be formalized, as in Calvo (1988),[15] is with multiple interest rate functions $R(b)$, selected from the correspondence defined in (1), which can be the low rate increasing schedule or the high rate decreasing one. Any other combination of those two schedules is also possible. The borrower is offered one schedule of the interest rate as a function of the debt level $b$ and chooses debt optimally given the schedule.

In summary, the assumption on the timing of moves is a key assumption to have multiple equilibria or a single equilibrium. If the creditors move first, there are multiple equilibrium interest rates and debt levels, and they are the same equilibria whether the borrower chooses current debt or debt at maturity. Instead, if the borrower moves first and chooses debt at maturity, as in Aguiar and Gopinath (2006) and Arellano (2008), there is a single equilibrium. Choosing debt at maturity amounts to picking the probability of default and therefore the interest rate as well. Finally, if the borrower moves first and chooses the current level of debt, given an interest rate schedule defined as a one-to-one mapping from $b$ to $R$, then the equilibrium will depend on the schedule and there is a continuum of equilibrium schedules. This is the approach in Calvo (1988).

**Lorenzoni and Werning (2013)** Lorenzoni and Werning (2013) use a dynamic, simplified version of the Calvo (1988) model, in which the borrower is a government with exogenous deficits or surpluses. In a two-period version, there is an exogenous deficit in the first period $-s^h$, with $s^h > 0$. In the second period, the surplus is stochastic, $s \in \left[-s^h, S\right]$, with density $f(s)$ and corresponding cdf $F(s)$. In order to finance the deficit in the first period, the government needs to borrow $b = s^h$. In the second period, it is possible to pay back the debt if $s \geq bR$, where $R$ is the gross interest rate charged by foreign lenders.

The creditors are competitive, they must make zero profits. It follows that $R^* = R\left(1 - F\left(bR\right)\right)$. If we had written $q = \frac{1}{R}$ and $a = bR$, the condition would be $R^* =$

[14] Trivially, it is still possible to obtain uniqueness in the case in which the borrower faces the supply curve in $R$ and $b$ defined by (1). If the borrower picks $R$, then it is able to select the low rate equilibrium directly. That is essentially what happens when the borrower faces the schedule $R(a)$ and picks $a$.

[15] In Calvo (1988), debt is exogenous.

$\frac{1}{q}(1 - F(a))$. As before, it is possible to use these equations to obtain functions $R(b)$ using the first equation and $q(a)$ using the second equation. These would be the two classes of schedules that were identified in the analysis earlier, when the government moves first. For the normal distribution, the schedules $R(b)$ and $q(a)$ will look like the supply curves in Figure 6. There are multiple equilibrium schedules $R(b)$. There is the good, increasing schedule and the bad, decreasing schedule, and there is a continuum of other schedules with points from any of those two schedules. The government that borrows $b = s^h$ may have to pay high or low $a = R(b)\, b$ depending on which schedule is being used with the corresponding probabilities of default.

What if the schedule, instead, is $q(a)$? The schedule is unique, but there are multiple points in the schedule that finance $b$. The government that borrows $q(a)\, a = s^h$ can do so with low $a$ and low $\frac{1}{q}$ or with high $a$ and high $\frac{1}{q}$. If the government is able to pick $a$, then implicitly it is picking the interest rate. Lorenzoni and Werning (2013) use an interesting argument for the inability of the government to pick the debt level $a$. For that they devise a game in which they divide the period into an infinite number of subperiods and do not allow for commitment in reissuing debt within the period. In that model, the government takes the price as given. The intuition is similar to the durable good monopoly result. In our model, the large agent also takes the price as given because of the timing assumption.

**Eaton-Gersovitz (1981)** In the model in Eaton and Gersovitz (1981), the borrower moves first, so it is key whether the equilibrium schedule is in $b$ or $a$. In our notation, they consider a schedule in $b$, $R(b)$. To be more precise, they define $R^*(b) = R(b)\, b$. Their equation (8) can be written as

$$[1 - \lambda(R^*(b))]\, R^*(b) = (1 + r^*)b$$

, where $\lambda$ is the probability of default that depends on the level of debt at maturity and $r^*$ is the risk free net interest rate. In our notation this can be written as

$$[1 - \lambda(R(b)\, b)]\, R(b) = R^*,$$

where $\lambda(R(b)\, b) \equiv F(1 + bR(b))$, which is equation (1) in our model. As seen earlier, there are multiple schedules in this case. Eaton and Gersovitz do not consider the decreasing schedule by assuming that $R(b)\, b$ cannot go down when $b$ goes up. This amounts to excluding decreasing schedules by assumption.[16]

## 4 Concluding remarks

In models with sovereign debt, interest rates are high when default probabilities are high. The object of this paper is to investigate conditions under which the reverse is also true, that default probabilities are high because interest rates are high. This means that there can be equilibrium outcomes in which interest rates are unnecessarily high and in which policy arrangements can bring them down. This exploration is motivated by the recent sovereign debt crisis in Europe, but it is also motivated by a literature that does not seem to be consensual in this respect. Indeed, although Eaton and Gersovitz (1981) claim that there is a single equilibrium, Calvo (1988), using a similar structure, shows that there are both high and low interest rate equilibrium schedules. Aguiar and Gopinath (2006) and Arellano (2008), building on Eaton and Gersovitz, modify an important assumption on the choice of debt by the large player and find a single equilibrium. We show that small changes in timing assumptions and actions of agents can explain these conflicting results. Part of the analysis is related to Lorenzoni and Werning (2015).

Assumptions on whether the country chooses the debt net of interest payments or gross of those payments, or whether the borrower moves first or the creditors do, are not assumptions that can be obtained directly from empirical evidence. But there is indirect evidence. The multiplicity of equilibria that arises under some of those assumptions is consistent with the large and abrupt movements in interest rates that are observed in sovereign debt crises, whereas the single equilibrium is not.

With our timing assumptions, there are both high and low interest rate equilibria. However some of those high interest rate equilibria, e. g. the ones in Calvo (1988), are fragile to simple refinements. That is no longer the case if the stochastic process for output is bimodal, meaning that with relatively high probability output can be either very high or very low. The empirical content of this is in the observation of long periods of growth followed by long stagnations that can be found in data for many countries. In the unraveling of the sovereign debt crisis in Europe, for some of the countries particularly exposed to it, such as Portugal or Italy, the hypothesis that

[16] See proof of Theorem 3 in Eaton and Gersovitz (1981).

future growth was drawn from a regime switching model with high and low growth, with high persistence, could not be rejected. Ayres et al. (2015) pursue this line in a quantitative exercise.

The high interest rate equilibria can be ruled out with policies of large purchases of sovereign debt, at penalty rates, in the spirit of the ones announced by the ECB back in 2012. Those policies could have the effect that they seem to have had, of bringing down sovereign debt spreads. According to this view, the ruling of the German constitutional court in early 2014, which "found that the central bank had overstepped its mandate and that OMT was a back door to 'monetary financing' of governments outlawed under European treaties," is unfounded.[17]

[17] "The German courts and the ECB: It isn't over," *The Economist*, February 15th, 2014.

# References


[1] Aguiar, M., M. Amador, E. Farhi, and G. Gopinath. 2014. "Sovereign Debt Booms in Monetary Unions." American Economic Review, 104(5): 101-06.

[2] Aguiar, M. and G. Gopinath. 2006. "Defaultable Debt, Interest Rates and the Current Account," Journal of International Economics 69, 64–83.

[3] Arellano, C. 2008. "Default Risk and Income Fluctuations in Emerging Economies," American Economic Review 98, 690–712.

[4] Ayres, J., G. Navarro, J. P. Nicolini, and P. Teles, 2015, The Role of Expectations in Sovereign Debt Crises: A Quantitative Exercise, mimeo, FRB Minneapolis.

[5] Bassetto, M. 2005. "Equilibrium and Government Commitment," Journal of Economic Theory 124, 79–105.

[6] Bocola, L. and A. Dovis. 2015. "Self Fulfilling Debt Crises: A Quantitative Analysis." Mimeo, Northwestern University.

[7] Calvo, G. A. 1988. "Servicing the Debt: The Role of Expectations," American Economic Review 78, 647–661.

[8] Cole, H. L. and T. J. Kehoe. 2000. "Self-Fulfilling Debt Crises," The Review of Economic Studies 67, 91–116.

[9] Conessa, J. C. and T. Kehoe. 2012. "Gambling for Redemption and Self-Fulfilling Debt Crises." Mimeo. FRB of Minneapolis.

[10] Corsetti, G. and L. Dedola. 2013. "The mystery of the printing press: Monetary Policy and Self-Fulfilling Debt Crises." Mimeo, Cambridge University.

[11] Eaton, J. and M. Gersovitz. 1981. "Debt with Potential Repudiation: Theoretical and Empirical Analysis," Review of Economic Studies 48, 289–309.

[12] Hamilton, J. D. 1989. "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle," Econometrica 57, 357–384.

[13] Jones, B. F. and B. A. Olken. 2008. "The Anatomy of Start-Stop Growth," Review of Economics and Statistics 90, 582–587.

[14] Lorenzoni, G. and I. Werning. 2013. "Slow Moving Debt Crises," mimeo, MIT.

[15] Roch, F. and H. Uhlig. 2015. "The Dynamics of Sovereign Debt Crises and Bailouts."

[16] Mimeo, The University of Chicago.

# Appendix 1

In this appendix, we perturb the game described in Section 2 by introducing a second stage at the end of the first period that allows for partial renegotiation. We then explore the robustness of the equilibria described in Section 2 to "very small" perturbations, in a sense we make precise later on.

Specifically, and given any outcome $(b, R)$ in the first stage of period 1, nature allows the borrower, with probability $\pi \in (0, 1)$, to make a take-it-or-leave-it offer of an alternative interest rate equal to $R - \delta$ to a coalition of a fraction $\alpha$ of lenders where $\delta > 0$ is exogenously given.[18] The coalition then chooses to accept the new rate or to keep the one in the first stage. Period 2 is the same as before: given the amount owed by the borrower and the realization of the endowment shock, the borrower decides to default or pay the debt in full. The payoff following default is as before.

We denote by $\alpha \in (0, 1)$ the measure of lenders that may be called into the coalition. It is important to emphasize that in this second stage, it is the coalition rather than each individual lender making decisions. Each agent in the coalition is treated equally. It is this assumption that imposes an $\alpha$-limited degree of coordination. As $\alpha \to 0$, there is no degree of coordination, and as we will show, the refinement requires $\alpha$ to be strictly bounded above zero. The first stage is exactly as before: all lenders—the $\alpha$ ones that can be called into the coalition and $(1 - \alpha)$ who cannot—then compete among each other, so they all charge the same rate in the first stage.[19]

Let this perturbed game be denoted by $G^{\alpha}(\delta, \pi)$.[20] We first characterize equilibria in the games $G^{\alpha}(\pi, \delta)$. In the spirit of trembling-hand perfection, we explore, given $\alpha$, which of the equilibria described in Section 2 are the limit of the sequence of equilibria of the games $G^{\alpha}(\pi, \delta)$ when $\pi \to 0, \delta \to 0$.

[18] Considering only reductions in interest rates is without loss of generality. If the borrower had the option of choosing higher interest rates, he would never do so.

[19] Note that the perturbation introduces ex ante heterogeneity. We will focus on the limiting cases where $\delta \to 0$ and $\pi \to 0$, so the heterogeneity is vanishing in the limit.

[20] The original game is equivalent to $G^{\alpha}(\delta, 0)$ or $G^{\alpha}(0, \pi)$.

**Definition 2** *Given $\alpha \in (0,1)$, an equilibrium $(R,b)$ in the game $G^{\alpha}(0,0)$ is robust to an $\alpha-$degree of coordination if it is the limit of the sequence of equilibria in the games $G^{\alpha}(\delta,\pi)$ when $\delta \to 0$ and $\pi \to 0$.*

We will show that the equilibria in the decreasing part of the zero-profit schedule

$$R^{*} \equiv R(b)\left[1-F\left(1+bR(b)\right)\right] \tag{7}$$

do not survive a refinement based on this perturbation, whereas equilibria in the increasing part do. Two assumptions are key to obtaining the results. First, the auction in the first stage must be anonymous (Lemma 1).[21] Second, a strictly positive degree of coordination of lenders is required (Result 2).

We prove the results in a series of steps. First, we show that, as long as the auction in the first stage is anonymous, there is no equilibrium in the perturbed game in which the offer is accepted.

**Lemma 1** *For any equilibrium of the perturbed game in which nature allows the borrower to make the offer $R-\delta$, the offer is rejected by the coalition if the auction in the first stage of the game is anonymous.*

**Proof.** Assume there is an equilibrium with $\delta > 0$ where the offer is accepted. The $\alpha-$members of the coalition get an interest rate of $R^c$ with probability $\pi$ and an interest rate of $R^c-\delta$ with probability $(1-\pi)$, whereas the lenders that cannot be part of the coalition get an interest rate of $R^n$. The expected return for lenders within the coalition is

$$\begin{aligned} R^{*} &= (1-\pi)R^{c}[1-F(1+b\left[\alpha R^{c}+(1-\alpha)R^{n}\right])] \\ &\quad +\pi(R^{c}-\delta)[1-F(1+b[\alpha R^{c}+(1-\alpha)R^{n}-\alpha\delta])], \end{aligned} \tag{8}$$

whereas the condition for the $(1-\alpha)$ fraction of agents that do not get to be part of the coalition is

$$\begin{aligned} R^{*} &= (1-\pi)R^{n}[1-F(1+b\left[\alpha R^{c}+(1-\alpha)R^{n}\right])] \\ &\quad +\pi R^{n}[1-F(1+b[\alpha R^{c}+(1-\alpha)R^{n}-\alpha\delta])]. \end{aligned} \tag{9}$$

[21] Anonymity is irrelevant for the game in Section 2, where all agents are homogeneous. But the perturbation introduces heterogeneity, so anonymity is important in the perturbed game.

As $\delta > 0$, $R^n < R^c$. It is immediate that this cannot be an equilibrium in a single-price auction, where $R^n = R^c$. Consider now a multi-price auction, where each lender receives the interest rate offered. Note that for the borrower, borrowing from each lender type implies the same expected payment. However, the lenders not in the coalition imply a payment with certainty, so the borrower will choose those lenders first. Thus, a fraction $(1-\alpha)$ receives an interest rate of $R^n$ and the fraction $\alpha$ receives the interest rate $R^c > R^n$. It immediately follows that in an anonymous auction, the noncoalition lenders best interest is to offer their funds at the rate $R^c$, where profits are higher than $R^*$. ■

If the auction is not anonymous, the borrower can fully discriminate in a multi-price auction and conditions (8) and (9) fully characterize the equilibrium interest rates. Once the borrower cannot discriminate the lender's type, the single-price auction provides incentives for truthful revelation but implies a unique interest rate, which breaks down the equilibrium. On the other hand, in a multi-price auction, agents not in the coalition do not have incentives to reveal their type, which also breaks the proposed equilibrium.

Note also that the perturbation we consider is a simple one in which agents know ex ante if they belong in the potential coalition or not. A more general perturbation would allow for each lender to have a probability $\alpha(j)$ of belonging to the coalition, with $\int_0^b \alpha(j)dj = \alpha$. The only case in which there can be an equilibrium with an anonymous auction in which an offer is accepted is the knife-edge case in which $\alpha(j) = \alpha$ for all $j$, so (9) is not an equilibrium condition anymore. In this case all agents are ex ante identical and anonymity plays no role.

We now characterize the conditions under which an offer will be accepted by the coalition for $\delta$ small enough.

Let the outcome of the first stage be a point in the schedule $(R, b)$, defined by (7). Assume that nature allows the borrower to offer $R - \delta$ to the coalition. If it accepts the new rate, their return will be given by

$$(R-\delta)\left[1 - F\left(1 + (1-\alpha)bR + \alpha b\left(R - \delta\right)\right)\right] \equiv E(\delta).$$

The new rate reduces the payment in case of no default, but it reduces the probability of default, so the net effect depends on which effect dominates. We now find a condition such that the second effect dominates, so reductions in the interest rates (positive values for $\delta$) can increase the expected payment to the coalition for small enough perturbations, so $\delta$ is close to zero. For that, we derive the expression above $E(\delta)$ with respect to $\delta$ and evaluate it at $\delta = 0$ to obtain

$$E'(\delta = 0) = -\left[1 - F\left(1 + b\right)\right] + R[f\left(1 + bR\right)\alpha b].$$

This expression is positive when

$$Rb\frac{f\left(1 + bR\right)}{\left[1 - F\left(1 + b\right)\right]} \equiv H(R, b) > \frac{1}{\alpha},$$

in which case the coalition would accept the offer for $\delta$ small enough.[22] We now show the results mentioned above.

**Result 1:** *If the pair $(R_1, b_1)$ is an equilibrium in the original game and is in the increasing part of the schedule defined by (7), then it is robust to an $\alpha-$degree of coordination for any $\alpha$.*

**Proof.** Differencing the identity (7) with respect to $b$, we obtain

$$R^{'}(b) = \frac{R^2 f(1 + bR)}{[[1 - F(1 + bR)] - Rbf(1 + bR)]}.$$

If $(R_1, b_1)$ is in the increasing part of the schedule, $R^{'}(b_1) > 0$. Since the numerator is positive, this implies that the denominator must be negative, which implies that

$$R_1 b_1 \frac{f(1 + b_1 R_1)}{[1 - F(1 + b_1 R_1)]} \equiv H(R_1, b_1) < 1.$$

Thus, $H(R_1, b_1) < 1 < \frac{1}{\alpha}$ for any $\alpha \in (0, 1]$, which means that the offer is not accepted for any degree of coordination, for some $\delta_1$ that is small enough. This means that the equilibrium in the original game, $(R_1, b_1)$, is also an equilibrium in the perturbed game, $G^\alpha(\pi, \delta)$, for any $\alpha > 0$, any $\pi$, and any $\delta < \delta_{1.}$ It therefore follows that $(R_1, b_1)$ is the limit of this sequence of games when $\delta \to 0, \pi \to 0$. ■

**Result 2:** *If the pair $(R_2, b_2)$ is an equilibrium in the original game and is in the decreasing part of the schedule defined by (7), then it is not robust to an $\alpha-$degree*

[22]Note that the smaller the coalition, the strongest is this condition. This is why the coalition is important.

*of coordination for any* $\alpha > \alpha^{\min}$, *where* $\alpha^{\min} < 1$ *is the minimal required degree of coordination.*

**Proof.** If $(R_2, b_2)$ is in the decreasing part of the schedule, $R'(b_2) < 0$. Since the numerator is positive,

$$R_2 b_2 \frac{f(1 + b_2 R_2)}{[1 - F(1 + b_2 R_2)]} \equiv H(R_2, b_2) > 1.$$

Let

$$\alpha^{\min} = \frac{1}{H(R_2, b_2)} < 1.$$

Assume that $(R_2, b_2)$ is robust to a $\alpha^{\min}-$ degree of coordination. This means that there is an equilibrium arbitrarily close to $(R_2, b_2)$ in the game $G^{\alpha}(\pi, \delta)$ for $\alpha > \alpha^{\min}$ and small enough values for $\pi$ and $\delta$. By continuity of the function $H(R, b)$, it follows that if nature lets the borrower make an offer $R_2 - \delta$, it will be accepted by a coalition larger than $\alpha^{\min}$, which contradicts Lemma 1. ■

Note that the value of $\alpha^{\min}$ is related to the value of $H(R_2, b_2)$ relative to 1. When the slope of the schedule defined by (7) becomes very close to $-\infty$, which happens when the equilibria in the decreasing schedule get arbitrarily close to the equilibria in the increasing schedule, $H(R_2, b_2) \to 1$ and $\alpha^{\min} \to 1$, requiring an arbitrarily large degree of coordination.

# 5 Appendix 2

In this Appendix we analyze the example of a discrete distribution,[23] corresponding to the bimodal distribution with zero standard deviation, $\sigma = 0$. The endowment process is

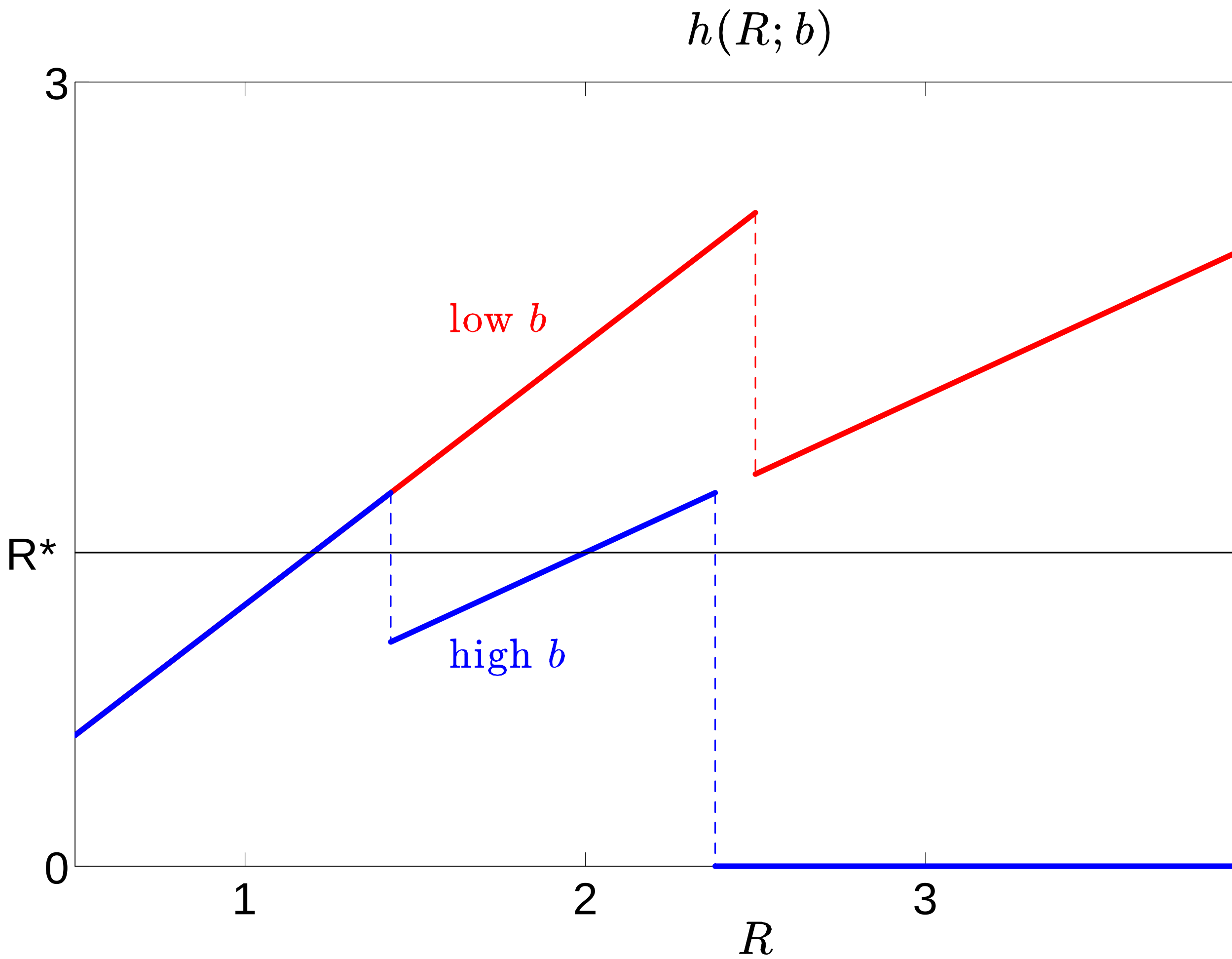
$$\begin{aligned} y_1 &= 1 \\ y_2 &= \begin{cases} y^l, & \text{probability } p \\ y^h, & \text{probability } (1-p) \end{cases} \\ 1 &< y^l < y^h \end{aligned}$$

Given a value for $b$, the expected return for lenders is

$$h(R;b) = \begin{cases} R, & \text{if } Rb \leq y^l - 1 \\ R\,(1-p), & \text{if } y^l - 1 < bR \leq y^h - 1 \\ 0, & \text{if } Rb > y^h - 1. \end{cases}$$

which can be plotted as

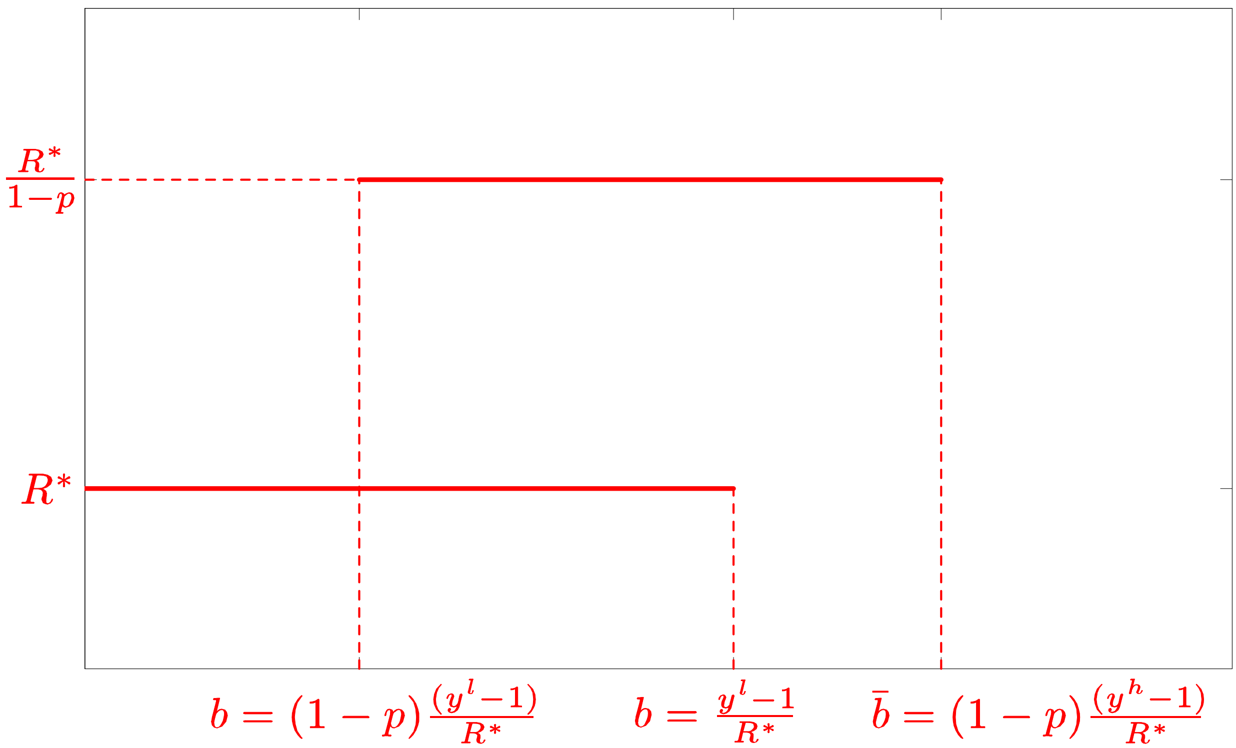
[23] We thank Fernando Alvarez for suggesting this example at the Banco de Portugal Conference on Monetary Economics, June 2015.




The supply curve by creditors is then

$$R(b) = \begin{cases} R^*, & \text{if } R^*b \leq y^l - 1 \\ \\ \frac{R^*}{1-p}, & \text{if } y^l - 1 < \frac{R^*}{1-p}b \leq y^h - 1 \\ \\ \infty, & \text{if } \frac{R^*}{1-p}b > y^h - 1 \end{cases}$$

plotted as

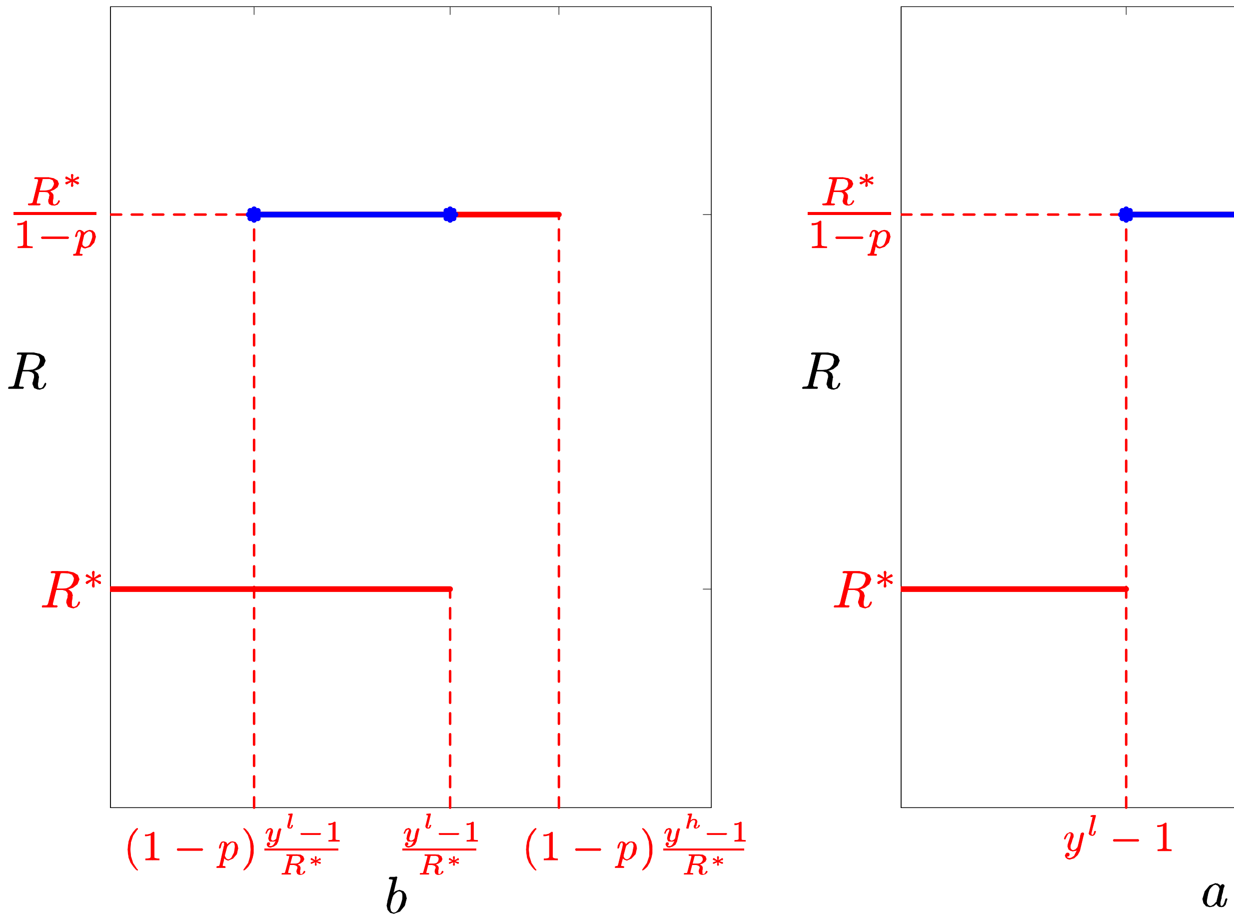



that, given a high enough demand, there may be two equilibria.

The same schedule, but in $R$ and $a = Rb$ is

$$R(a) = \begin{cases} R^*, & \text{if } a \leq \left(y^l - 1\right) \\ \frac{R^*}{1-p}, & \text{if } \left(y^l - 1\right) < a \leq \left(y^h - 1\right) \\ \infty, & \text{if } a > \left(y^h - 1\right) \end{cases}$$

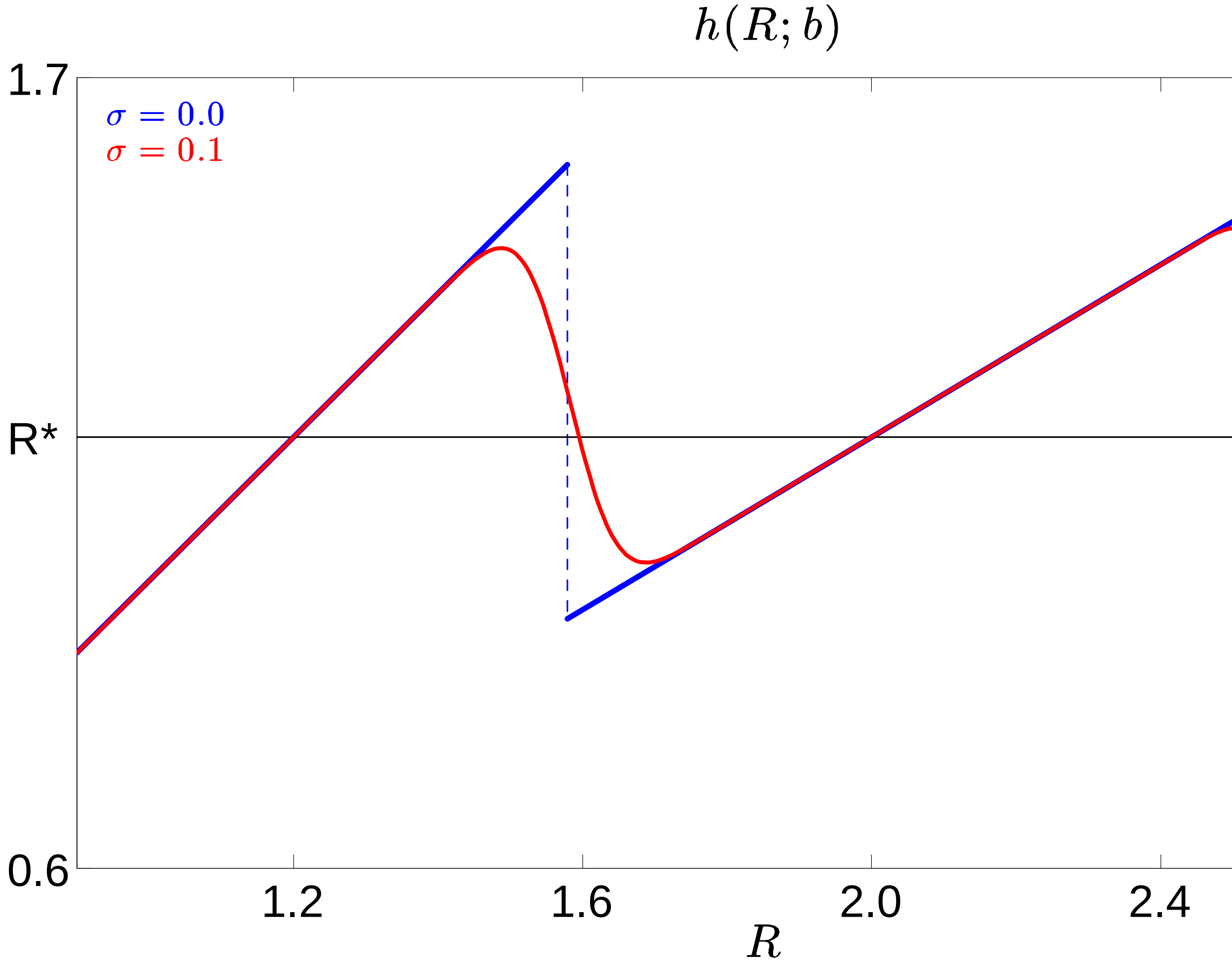
The two schedules can be plotted as



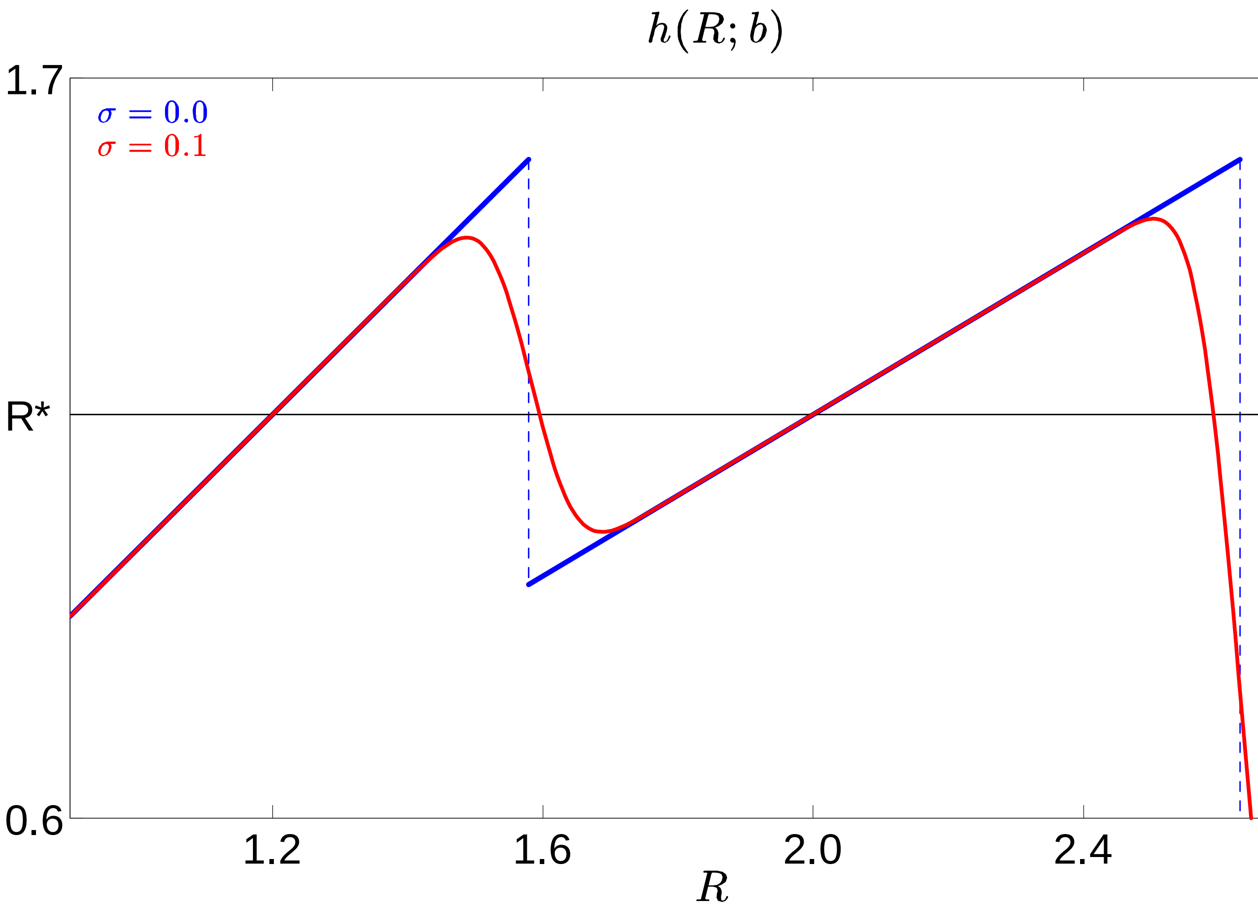

the blue segment corresponds to the unnecessarily high rates of interest.

If the borrower were to move first, would be able to pick a point on the schedule. Facing a schedule for debt at maturity $a$, the borrower would not pick a point on the blue part of the schedule. In that part of the schedule for a given $b$, because of the high rates, $a$ is higher than it is needed be. There is a point on the lower (red) part of the schedule where for the same $b$, $a$ is lower. If, instead, the schedule was a schedule for $b$, then there are tow possible schedules, and once offered the high rate schedule, for a given $b$, the borrower has no alternative but to take the high rate.

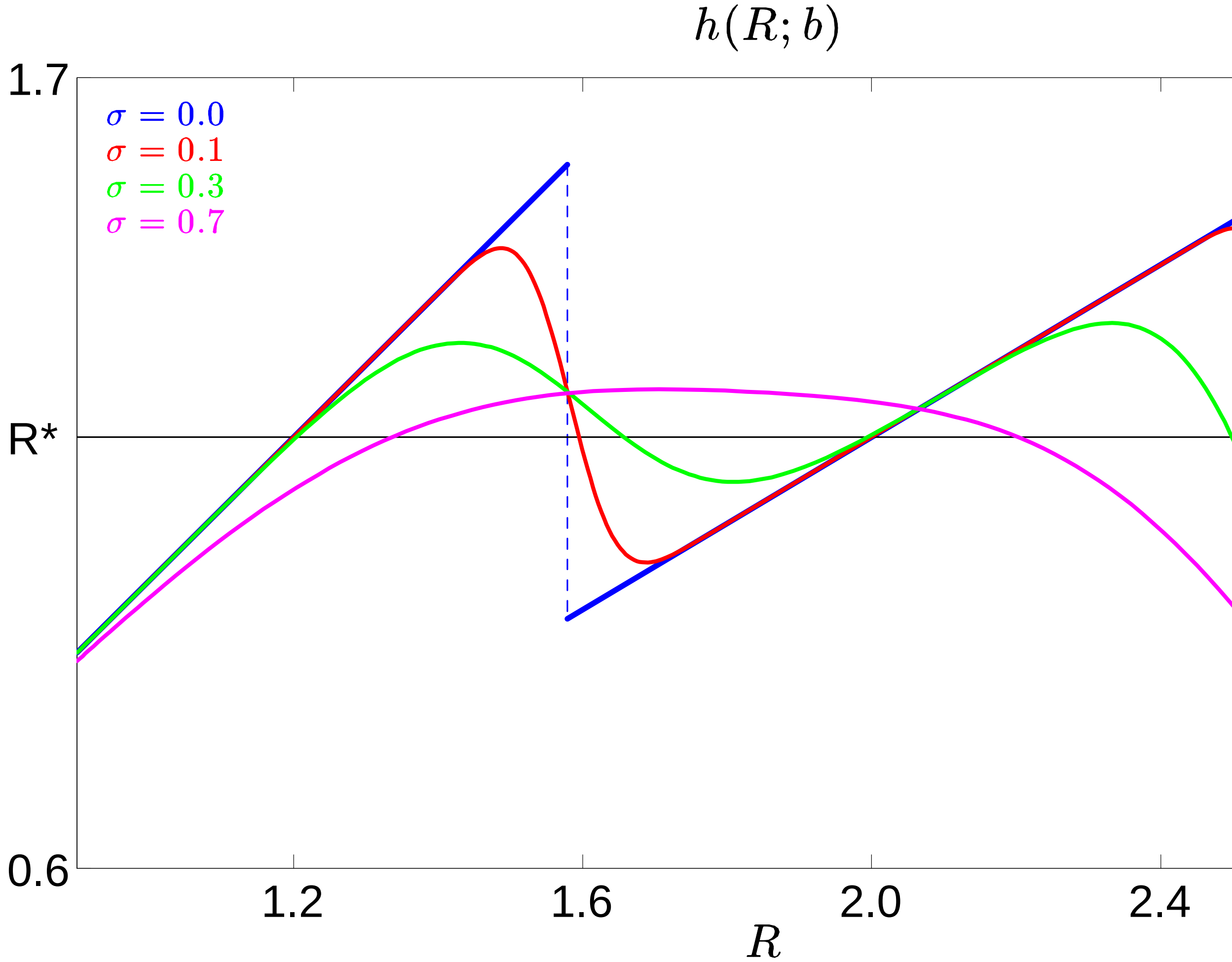
In this case of a discrete distribution, there are no decreasing schedules. They will be there for positive standard deviation as the following plot suggests.



$h(R;b)$
1.7
$\sigma = 0.0$
$\sigma = 0.1$
R*
0.6
1.2
1.6
2.0
2.4
$R$



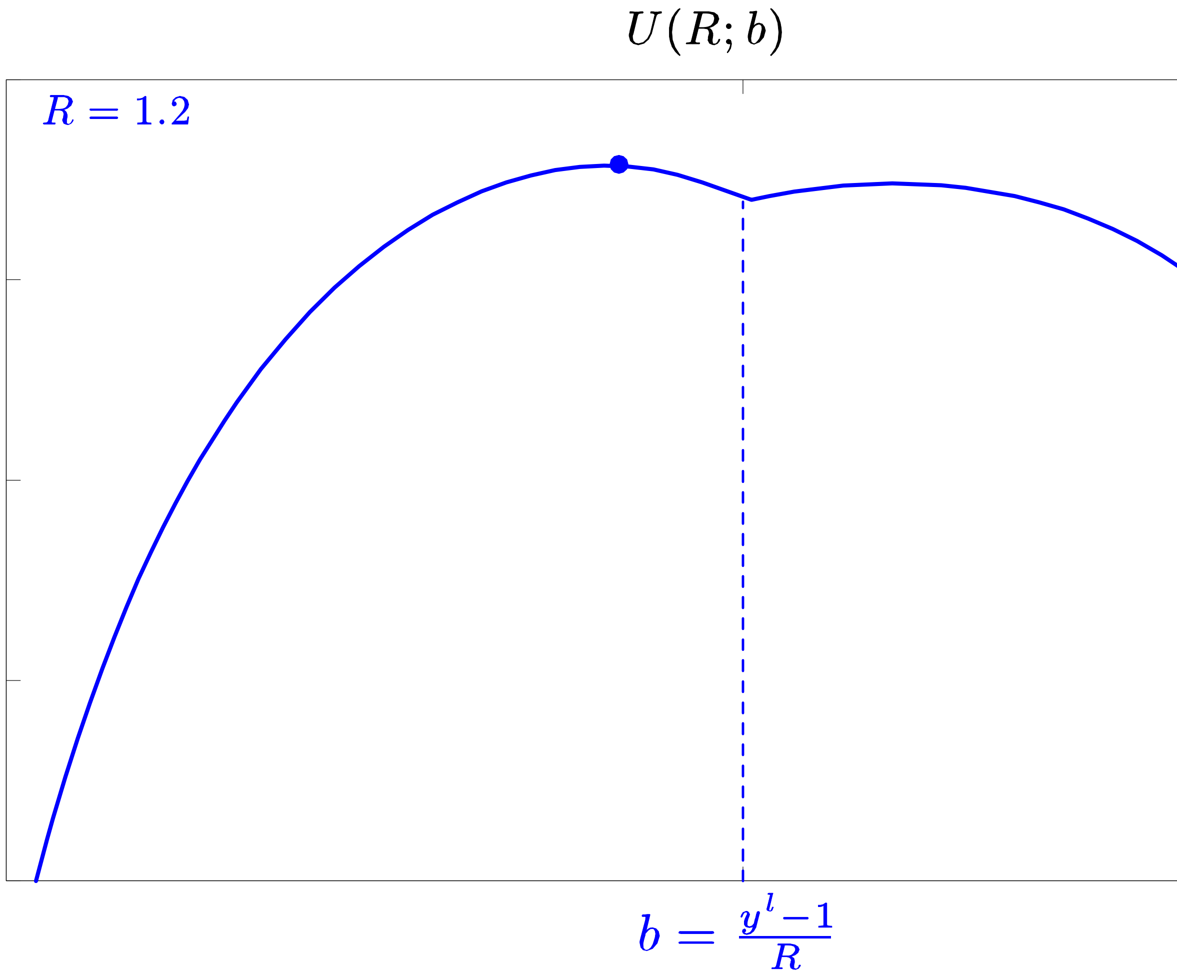

For a standard deviation that is big enough, there are no longer multiple increasing schedules, as seen in the following plot.



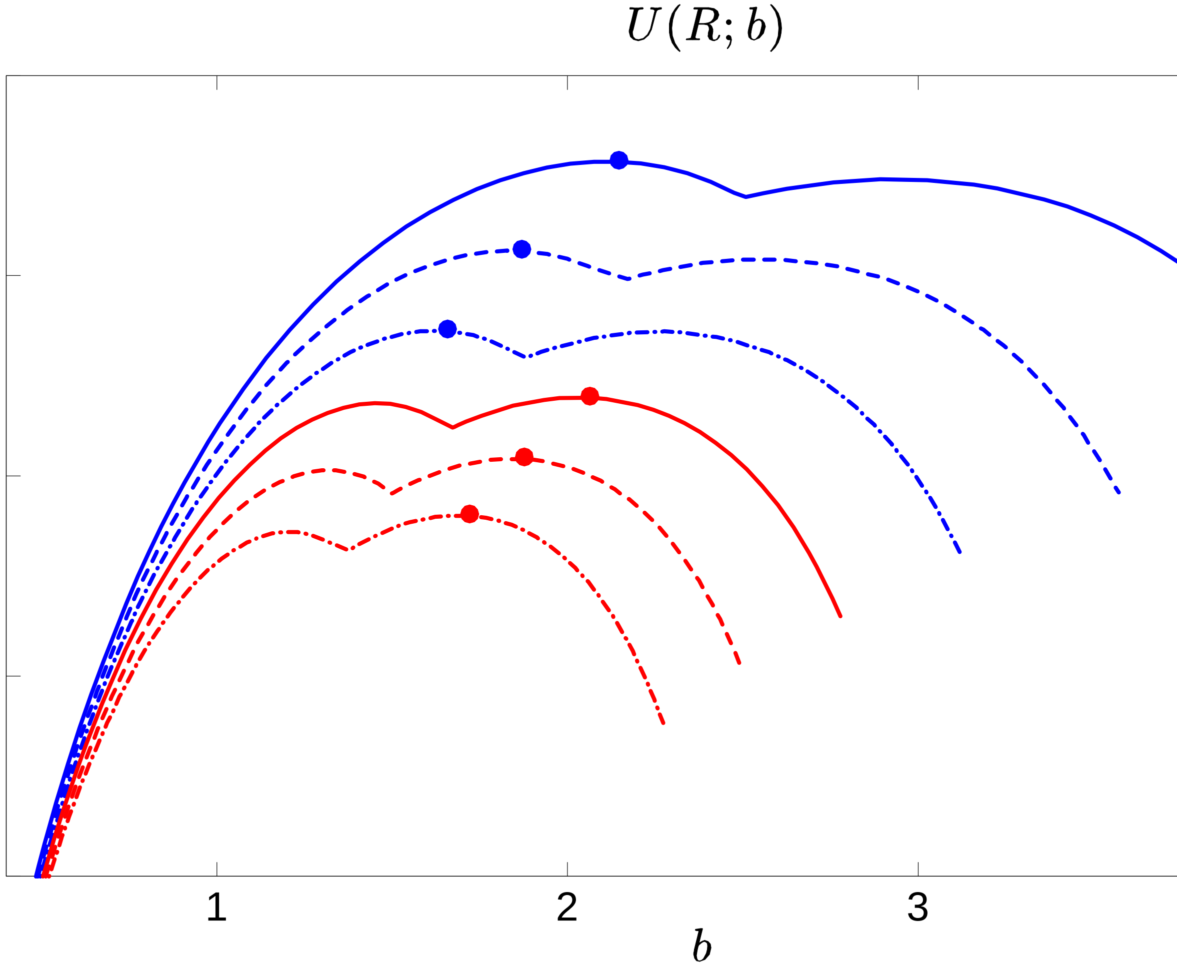

In order to compute an equilibria, we need to derive the demand for debt. Given the interest rate $R$, the borrower chooses $b$ in order to maximize the objective function

$$W(b) = \begin{cases} \begin{array}{l} U(1+b) + \beta p U\left(y^l - Rb\right) \\ +\beta(1-p)U\left(y^h - Rb\right) \end{array}, & \text{if } b \leq y^l - 1 \\ \\ \begin{array}{l} U(1+b) + \beta p U(1) \\ +\beta(1-p)U\left(y^h - Rb\right) \end{array}, & \text{if } y^l - 1 < b \leq y^h - 1 \\ \\ U(1+b) + \beta U(1), & \text{if } b > y^h - 1 \end{cases}$$

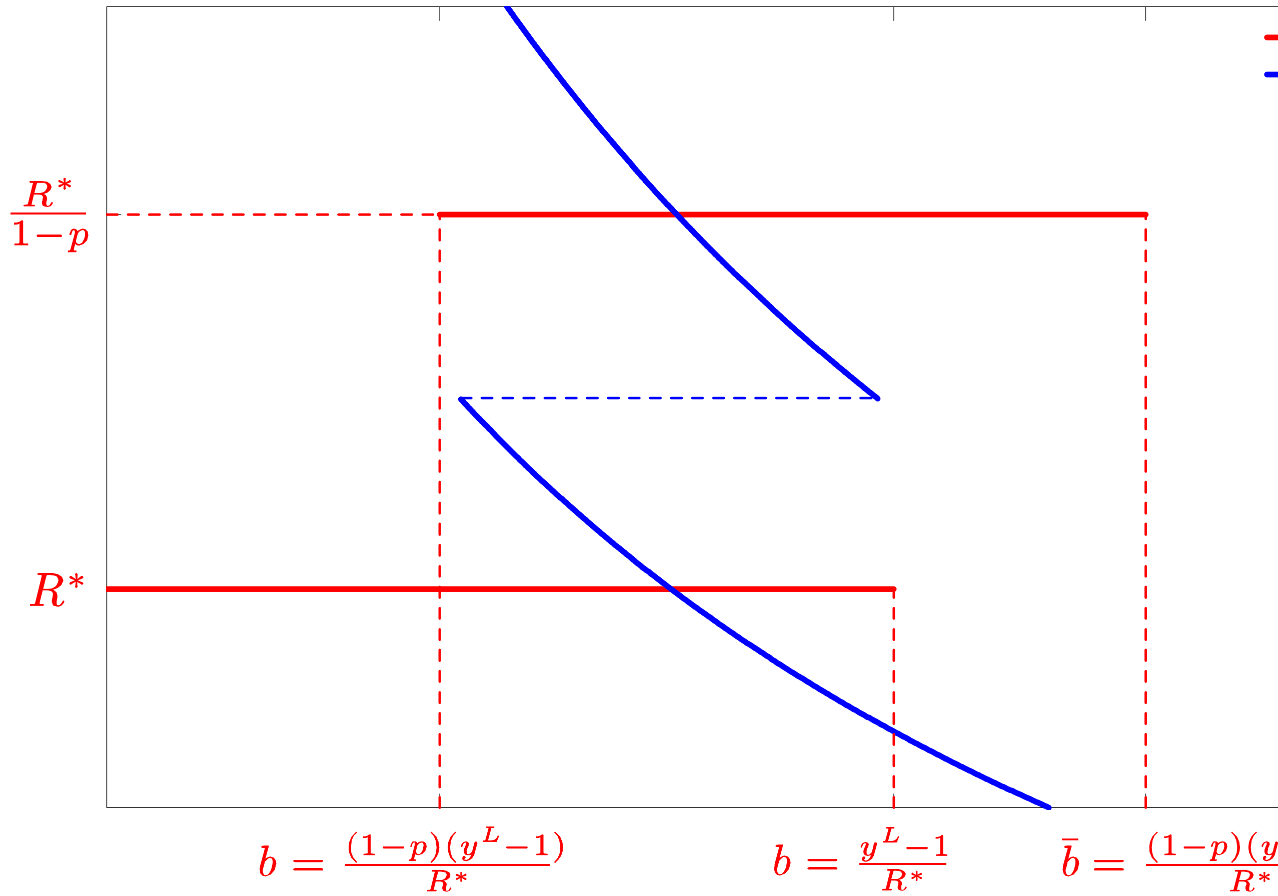
The objective can be drawn as




There is a convexity, because for a given interest rate $R$ the higher is the level of debt the lower is the probability of repayment. For a given $R$, higher debt is less expensive. As the interest rate goes up, the value goes down, but the effect is larger for lower levels of debt. For very high levels of debt, the interest rate does not matter, because debt will not be paid back with probability one. The implication of this is that while the maximum of the objective if with relatively low debt for relatively low interest rates, when interest rates increase, the optimal level of debt jumps up.




Below is the plot with the supply and demand, with multiple equilibria for the interest rate.




## Appendix 3

In this appendix, we show how small perturbations to the uniform distribution can give rise to multiple equilibria of the type obtained with the bimodal distribution.[24]

In the case of the uniform distribution, it is straightforward to obtain the solutions of $R^* = h(R;b)$, so that the supply curve can be described analytically. Let the distribution of the endowment process be the uniform, $f(y) = \frac{1}{Y-1}$, so that $F(y) = \frac{y-1}{Y-1}$. Then, from (1), the equilibrium interest rates must satisfy

$$R = \frac{1 \pm \left(1 - 4\frac{R^* b}{Y-1}\right)^{\frac{1}{2}}}{2\frac{b}{Y-1}},$$

[24]The uniform distribution is used only as an example.

provided $1 - 4\frac{R^* b}{Y-1} \geq 0$. The maximum level of debt consistent with an equilibrium with borrowing is given by $b^{\max} = \frac{Y-1}{4R^*}$. Below this value of debt, for each $b$, there are two possible levels of the interest rate.

Consider a perturbation $g(y)$ of the uniform distribution, so that the density would be $f(y) = \frac{1}{Y-1} + \gamma g(y)$, with $\int_1^Y g(y)dy = 0$. In particular, the function $g$ can be $g(y) = \sin ky$, with $k = \frac{2\pi}{Y-1}N$, where $N$ is a natural number. If $N = 0$, the distribution is uniform, so there is a single increasing schedule. If $N = 1$, there is a single full cycle added to the uniform distribution. The amplitude of the cycle (relative to the uniform distribution) is controlled by the parameter $\gamma$. The number of full cycles of the $\sin ky$ function added to the uniform is given by $N$. As $\gamma \to 0$, so does the perturbation.

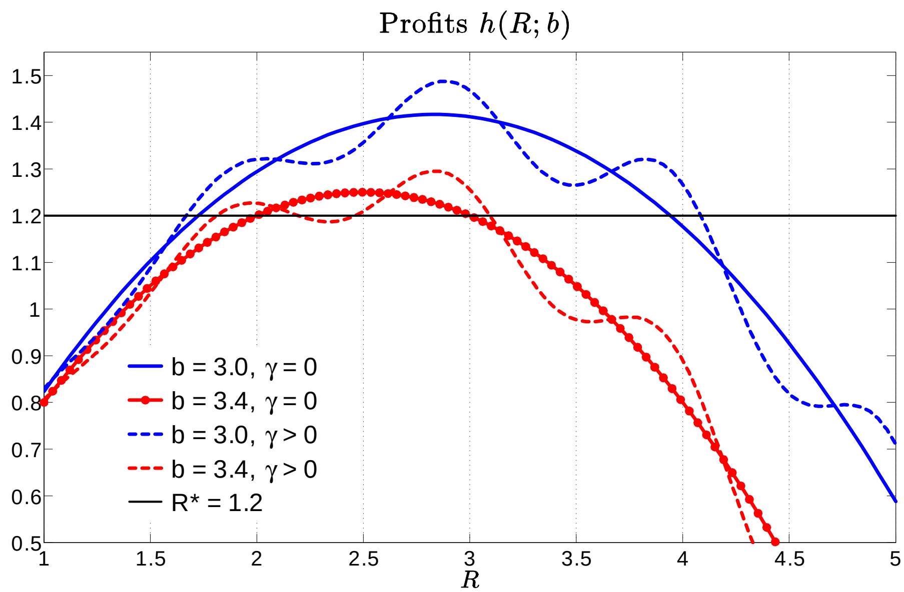



Figure 7: Perturbing the uniform distribution

Given a value for $\gamma$, the closer the debt to its maximum value, the larger the degree of multiplicity. The equation

$$\frac{1}{R} - \frac{1}{R^*}\left[1 - \frac{1+bR}{Y-1} - \gamma \sin kbR\right] = 0$$

has more than two solutions for $R$, for $\gamma$ that can be made arbitrarily small, as long as $b$ is close enough to $b^{\max}$. On the other hand, if $b$ is lower than $b^{\max}$, there is always a $\gamma > 0$ but small enough such that there are only two zeros to the function above.

An illustration is presented in Figure 12 for two levels of the debt and for two values of $\gamma$. As can be seen, when the debt is low, a positive value of $\gamma$ is not enough to generate multiplicity, but multiplicity arises as the level of the debt goes up. Note that if $\gamma$ is small, it may take a very long series to identify it in the data. Thus, it is hard to rule out this multiplicity based on calibrated versions of the distribution of output if the debt is close enough to its maximum.[25]

[25] This resembles the result in Cole and Kehoe (2000), where the fraction of short-term debt affects the chances of multiplicity.