Leaning against the wind in the New Keynesian model with heterogeneous expectations*

Mikhail Anufriev[a,b] Fabio Lamantia[b,c] Davide Radi[d,b]
Tomas Tichy[b]

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[a] *Economics Discipline Group, Business School, University of Technology Sydney*
[b] *Department of Finance, VŠB - Technical University of Ostrava, Ostrava*
[c] *Department of Economics, Statistics and Finance, University of Calabria, Rende*
[d] *DiMSEFA, Catholic University of Sacred Heart, Milan*

Abstract

In this paper, we explore the efficiency of the Leaning Against the Wind (LAW) policy within the New Keynesian framework with heterogeneous expectations. To do so, we introduce the financial sector into the benchmark model using the financial accelerator channel. We show that the LAW policy can be counterproductive because it creates an equilibrium that can coexist with the desirable equilibrium. A more stringent LAW policy reduces the stability region of the desirable equilibrium. However, in cases where the desirable equilibrium is unstable, the LAW policy can help reduce the amplitude of economic fluctuations.

Keywords: New Keynesian Model, Financial Market, Heterogeneous Expectations, Leaning Against the Wind, Complex Economic Systems, Economic Dynamics.

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1 Introduction

Central banks have a dual responsibility of maintaining price stability and promoting economic growth. To achieve this, they typically use monetary policy tools like interest rates and open market operations to control inflation and ensure the economy is operating at full capacity. However, asset price bubbles can threaten financial stability and economic growth, as seen in the notable crashes of 1929 and 2008. In such instances, when asset prices rise rapidly and create a bubble, its eventual burst may lead to a sharp decline in prices and potentially cause economic turmoil. Kindleberger (1989) and Reinhart and Rogoff (2009) provide detailed accounts of such events.

The question then arises whether central banks should monitor financial markets and intervene when bubbles emerge. The central bank's policy reacting directly to the state of the financial market is often referred to as a "*leaning against the wind*" (LAW) policy. The advantages and disadvantages of the LAW policy have been extensively debated in macro literature over the last 25 years. The rationale behind the policy is that by proactively addressing the bubble before it bursts, the monetary authority aims to prevent the potential contagion of a financial market crash to the real economy. However, there are several reasons why monetary policy *should not* respond to asset bubbles. First, determining the fundamental value of an asset and thus diagnosing a bubble is a challenging task. Furthermore, since asset markets evolve rapidly and monetary policy operates with a lag, there is a risk that the policy may be poorly timed. Additionally, as there are numerous assets, it is unclear which ones the policy should target. Bernanke (2010) reflects on U.S. monetary policy before the GFC and, using these arguments, suggests that the policy was not excessively accommodating despite the presence of a housing bubble. Overall, the concern is that the LAW policy may weaken the economy without achieving its intended goal.

In this paper, we examine the efficiency of the LAW policy question within a macroeconomic behavioral approach that emphasizes learning and heterogeneity of expectations, as discussed in Hommes (2021). We propose a model that connects the real economy and financial market and allows some instability to develop in both these sectors due to heterogeneity of expectations. We combine the financial and real sectors within one model, as there has been growing interest and success in modeling heterogeneous expectations in both these fields (Brock and Hommes, 1998; Hommes, 2013; Hommes and Lustenhouwer, 2019a). This approach is further supported by experimental evidence from "learning-to-forecast" experiments, which showed that heterogeneous expectations lead to fluctuations both in financial markets (Anufriev and Hommes, 2012) and in the real sector (Assenza et al., 2021). These experiments underscore the importance of allowing for non-rational expectations and endogenous bubbles in macroeconomic models. While recent experimental studies have investigated the effectiveness of monetary policy rules for stabilizing financial markets (see Bao and Zong, 2019 and Hennequin and Hommes, 2023), a theoretical contribution in this field was lacking, and we aim to fill this gap.

We use the New Keynesian model with heterogeneous expectations (Hommes and Lustenhouwer, 2019a,b) and add a simple financial market to it with one riskless and risky asset. We then connect this market with the real economy via the "financial accelerator" channel. That is, we assume that the state of the financial market affects the expectations about the output in the New Keynesian model, with a bullish market leading to higher expectations. This can be due to the wealth effect of consumers or easier credit conditions for firms, or both.[1] There is also a link in the opposite direction, from the real economy to the financial market, operating via the interest rate, which is set by the monetary authority using a forward-looking Taylor rule.

Our way to formalize expectations traces back to models of Brock and Hommes (1998) and De Grauwe (2011) that depart from the rational expectations assumption in the financial market and the real sector, respectively. We consider a large number of agents in both sectors who are aware of the target levels of inflation, output gap, and the fundamental value of the risky asset, but may deviate from these values due to some reason, such as strategic uncertainty, absence of trust to the monetary authority, etc. Notably, these agents monitor the values of these variables and use them to compute the fitness of different expectations, which they then use to update their expectations. Therefore, even though our agents are non-rational, they learn from the past. To make the model tractable analytically, we employ the large type limit idea presented in Brock et al. (2005) to reduce the system's dimensionality. This

[1] Bernanke and Gertler (1995) and Bernanke et al. (1996) put forward the balance sheet channel via which the financial markets' conditions impact the real economy, and argue for a resulting financial accelerator effect. A related approach stressing the role of the financial market for credit is investigated in Kiyotaki and Moore (1997) model.

approach is also used in other papers, such as Anufriev et al. (2013), Hommes and Lustenhouwer (2019b) and Pecora and Spelta (2017). Also, like the latter paper, we interpret the variance of expectations parameter as a measure of the population's "anchoring" to the central bank's targeted policy, which may reflect a degree of trust in the policymakers.

The monetary authority relies on the forward-looking Taylor rule, which takes into account expected inflation, expected output gap, and expected asset prices. While it is widely accepted that conditioning on inflation and output gap is important, we seek to assess the necessity and effectiveness of conditioning the Taylor rule on the financial market, referred to as the LAW policy. In the absence of sensitivity to the financial market and the accelerator effect, our real economy corresponds to the one studied in Hommes and Lustenhouwer (2019b). Therefore, all novel dynamic phenomena and differences arise due to the accelerator effect and LAW policy.

Our main findings are as follows: Firstly, we found that when the interaction between the real and financial sectors is mediated in both directions (i.e., when the financial accelerator is present and the interest rate affects the financial market), multiple equilibria exist in the system. Specifically, apart from the desirable equilibrium with targeted levels of inflation, output gap, and fundamental price, there exists an additional, *non-targeted equilibrium* where all these variables are at other levels. The financial market will have a constant but non-fundamental price, which will impact the real economy. Only one of the two equilibria can be stable. Secondly, the implementation of the LAW policy introduces a third equilibrium, referred to as the "LAW equilibrium". This equilibrium may be stable together with the desirable equilibrium, and in such cases, the LAW policy might have counterproductive effects as agents coordinate on this LAW equilibrium. Thirdly, with a more restrictive LAW policy, we observe that the stability region of the desirable equilibrium shrinks (an issue that can be addressed by conditioning the activation of the LAW policy on reaching a minimum level of mispricing). Simultaneously, the stability region of the LAW-equilibrium enlarges. Overall, we find that the LAW policy does not contribute to stabilizing the desirable equilibrium. However, fourthly, when the desirable equilibrium is unstable, leading to periodic fluctuations, the LAW policy proves effective in reducing the amplitude of these fluctuations, presenting a unique solution to this issue.

Literature Review

We contribute to the extensive body of work on studying the Taylor rule monetary policy, including the option to condition it on the state of the financial market (that is, the LAW policy), in the New Keynesian framework. Prior research, such as Clarida et al. (1998, 1999, 2000), has explored the empirical evidence of Taylor rules and their theoretical foundations. However, there remains a lack of consensus regarding the usefulness of the LAW policy. While some academic studies (Bernanke and Gertler, 2000, 2001) and policy-makers' reflections (Bernanke, 2010) propose that the uncertainty surrounding asset price distortions makes conditioning the rule on past prices impractical, others, such as Cecchetti et al. (2002), argue that central banks should identify and address bubbles based on size and source. The inefficiency of the LAW policy is emphasized in Svensson (2017), claiming that its costs outweigh the benefits significantly. However, Adrian and Liang (2018) contest this, suggesting that preventing financial vulnerability build-up, common under accommodative policies, offers benefits surpassing the costs. Overall, the literature indicates that the LAW policy may help prevent large bubbles, even without perfect identification. However, it should be applied carefully, as it is less effective and may even be counterproductive if it is too strong or if the actual reason for a surge in asset pricing is not correctly identified.

In contrast to all these studies, we investigate the LAW policy within a dynamic framework with boundedly rational agents. Recent research, surveyed in Hommes (2021), demonstrates that this framework is able to capture realistic deviations from the rational benchmark that occur in systems with positive expectation feedback, which are indeed present in the model connecting real and financial markets.

In our modeling we thus closely follow the literature that begins with the microfounded version of the New Keynesian model and replaces the aggregate expectations with heterogeneous expectations. This framework enables agents to dynamically choose their expectations using the discrete choice mechanism proposed in Brock and Hommes (1997). Axiomatic conditions that allow such a model to remain microfounded are derived in Branch and McGough (2009).[2] In the model with two

[2] Massaro (2013) and Kurz et al. (2013) derive general versions of the model with heterogeneous expectations. Hommes and Lustenhouwer (2019a) use the simplified version of the latter framework.

types of expectations, rational and extrapolative, Lines and Westerhoff (2010) find bounded but chaotic dynamics. Branch and McGough (2010) investigate the determinacy of the equilibrium in the model with rational and adaptive expectations. De Grauwe (2011) study dynamics with two opposed biased types and demonstrate the phenomenon of "animal spirits," which involves waves of persistent optimism or pessimism following shocks. In a model with two types of expectations, those that coincide with targets and naïve, Hommes and Lustenhouwer (2019a) examine the role of the endogenous credibility of a central bank, while Lustenhouwer (2021) explore the model with multiple biased types. The study in Hommes and Lustenhouwer (2019b), which is most closely related to our paper, analyzes the role of expectation anchoring in the model when the number of types goes to infinity. The paper demonstrates that the model can generate waves of optimism or pessimism when expectations are not anchored, and it investigates the effectiveness of monetary policy under this scenario, including the case of the zero lower bound. In contrast to all these papers, our model incorporates the financial sector connected to the real sector, allowing us to study the leaning-against-the-wind policy within this framework.

Our research is largely inspired by earlier models that incorporate both the financial and real sectors. For example, Westerhoff (2012) demonstrates the possibility of volatility contagion from the financial market to the real economy via the wealth effect. In comparison to this model, our model includes the impact of the real economy on the financial market, incorporates a richer set of expectations, and focuses on monetary policy. Models in De Grauwe and Macchiarelli (2015) and De Grauwe and Gerba (2018) extend the model of De Grauwe (2011) by incorporating the banking sector and financial frictions. These models show that animal spirit waves can be amplified due to pro-cyclical credit conditions. Although our model does not explicitly include the banking sector, it shares similarities in the way it connects the financial market with the real economy through the IS curve. Additionally, we differ by analyzing the model with multiple types and studying the effect of the Taylor rule policy. Lengnick and Wohltmann (2016) incorporate the financial market model of Westerhoff (2008) into the New Keynesian model of De Grauwe (2011), establishing four distinct channels (two of which are micro-founded by adding stock to agents' utility functions) that can either stabilize or destabilize the connected sectors. The authors aim to determine the optimal monetary policy, which may involve conventional (Taylor rule type) and unconventional (direct purchase of assets) policies. In contrast, we focus only on the two channels (one in each direction) that connect the sectors, enabling us to derive clearer analytical results. Furthermore, we address the effectiveness of the LAW policy by directly including asset prices in the Taylor rule.

Several experimental papers have studied stabilizing policies in the New Keynesian framework and thus address the same questions that we analyze here theoretically. For instance, Fischbacher et al. (2013) investigates the impact of monetary policy on the emergence and size of bubbles in the speculative asset market, similar to Smith et al. (1988), but with participants having the option to save with an interest-bearing bond. The study finds that increasing the interest rate reduces liquidity but has only a limited effect on the stock price. In the experiment by Galí et al. (2021), which has an overlapping generations structure, participants allocate their endowment between a risky stock and a riskless bond when young. The study shows that the LAW policy has an immediate desirable effect of reducing the bubble, followed by an opposite effect in the next period, thus highlighting the potential adverse effects of the policy. Bao and Zong (2019) and Hennequin and Hommes (2023) study the LAW policy within the learning-to-forecast experiments (LtFE), where participants make economic forecasts.[3] Contrary to the findings of Fischbacher et al. (2013) and Galí et al. (2021), they conclude that an interest rate policy can be highly effective in reducing bubbles, as long as the policy is sufficiently strong. None of these studies include a real sector. Some LtFE studies have investigated the New Keynesian economy under various monetary and fiscal policies, including the Taylor rule (Pfajfar and Žakelj, 2014; Arifovic and Petersen, 2017; Assenza et al., 2021; Kryvtsov and Petersen, 2021). These studies demonstrate that the monetary authority can achieve stability by reducing coordination on extrapolative expectations through a strong reaction to inflation. However, this policy may not be efficient if the economy becomes stuck in a recession and reaches the zero lower bound. However, none of these papers include a financial sector.

The only study that we are aware of that combines a real sector with an asset

[3]In the LtFE, participants do not trade and make no other economic decisions, but only submit predictions. The economic variables are consistent with rational strategies based on these predictions and market clearing. For the motivation and methodology of the LtFE, see Hommes (2011). Bao et al. (2013, 2017) and Arifovic et al. (2019) compare the LtFE with learning-to-optimize experiments where participants make economic decisions.

market is the experiment in Fenig et al. (2018). Participants submit labor supply, output demand, and asset trading decisions. A LAW policy unintentionally gives rise to bubbles at first, but rapidly increasing interest rates are successful in quickly deflating bubbles and stabilizing asset prices. The policy does not seem to have large negative effects on production.

The rest of the paper is organized as follows. In Section 2, we first describe the New Keynesian model for the real sector, then introduce the financial market, and finally apply the large type limit of Brock et al. (2005). Here we also formalize our way of connecting the financial market with the real economy via the financial accelerator. In Section 3, we study our model analytically. We build the analysis step by step, first by adding the financial accelerator and then the term in the interest rule responsible for the LAW policy. We simulate the model and illustrate the efficiency of the Taylor rule in our economy with real and monetary sections in Section 4. Section 5 provides final remarks and directions for future work. Technical derivations are presented in the Appendices.

2 A New Keynesian model with financial market

In this section, we present our model. Section 2.1 describes the log-linearized New Keynesian model with heterogeneous expectations, where we utilize the model of Hommes and Lustenhouwer (2019b). This enables us to rely on that paper when comparing policy analysis results in the presence of financial market effects. The model is consistent with a general monetary policy framework, as outlined in Woodford (2003) for a comprehensive treatment. Hommes and Lustenhouwer (2019a) derive this model with heterogeneous expectations based on utility-maximization principles.[4] Section 2.2 describes the canonical asset-pricing model with heterogeneous expectations, introduced in Brock and Hommes (1998). We employ this model for the financial market for two reasons. First, this model has undergone extensive study and has garnered substantial empirical support (Hommes, 2002; Boswijk et al., 2007;

[4]To incorporate heterogeneous expectations, they follow Kurz et al. (2013) and rely on the model-independent future choices of expectations made by different agents each period. The same model with heterogeneous expectations has also been used in the experimental study conducted by Assenza et al. (2021) and in the theoretical paper by Lustenhouwer (2021), among others.

Hommes and in 't Veld, 2017; Ter Ellen et al., 2021). Second, the setup of heterogeneous expectations in this model can be made similar to the way these expectations are constructed in the NK model. This similarity facilitates consistent analysis and enhances interpretability. We model expectations in Section 2.3, and the monetary policy in Section 2.4.

2.1 Real Sector

The reduced form of the macro-model is given by two linear equations: an investment-saving (IS) curve describing the dynamics of the output gap and a New Keynesian Phillips curve (NKPC) describing the dynamics of inflation. The dynamic IS curve reflects the traditional negative relationship between the real interest rate and the output (equal to consumption in this model) enhanced by the positive effect of expectations of future output on the current output arising from the households' motives to smooth consumption. The NKPC describes the positive association of inflation and the output gap, augmented with a positive effect of expected inflation on current inflation stemming from profit-maximizing firms' behavior. Log-linearized output gap, x_t, and inflation, π_t, in period t are given by:

$$\begin{aligned} x_t &= \bar{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \bar{r} - \bar{E}_t \pi_{t+1}\right) + \epsilon_t^d \\ \pi_t &= \beta \bar{E}_t \pi_{t+1} + \kappa x_t + \epsilon_t^s \,. \end{aligned} \tag{1}$$

In this model, individual subjective point expectations of the next period output gap and inflation are aggregated by simple averaging. Those averages are denoted as $\bar{E}_t x_{t+1}$ and $\bar{E}_t \pi_{t+1}$. We explain how these expectations evolve in Section 2.3.

The model depends on several behavioral parameters of consumers and firms. The parameter $\sigma > 0$ is an inverse of the households' elasticity of substitution of consumption over time. The larger σ is, the stronger marginal utility reacts to the change of consumption, and thus the less willing the household is to shift consumption over time. The parameter $\beta \in (0, 1)$ is the households' discount factor. The larger it is, the more patient the household is. Finally, the coefficient $\kappa > 0$ is defined as

$$\kappa = (\sigma + \eta)\frac{(1-\omega)(1-\beta\omega)}{\omega}\,, \tag{2}$$

and depends on two other parameters: η, an inverse of the households' elasticity to labor supply, and $\omega \in (0,1)$, the fraction of goods in the economy whose prices remain fixed each period, as in the model of Calvo (1983). Thus, ω measures the degree of firms' rigidity. The dynamics in (1) also depend on the deviation of the nominal interest rate, i_t, from the target level $\bar{r}$. It is assumed that the central bank uses the Taylor-type interest rate rule to set the nominal interest rate i_t. The rule is introduced in the next section, and in the paper, the dynamic properties of the model will be investigated as dependent on the parameters of the rule. As for the target real interest rate, as typical, we fix it at the steady-state level, that is, $\bar{r} = 1/\beta - 1$. Finally, ϵ_t^d and ϵ_t^s are demand and supply shocks, respectively. In the model with rational expectations they are modeled as AR(1) processes to calibrate the data (Clarida et al., 1999), but in our model, it will be sufficient to assume that they are independent white noise processes.

2.2 Financial Market

Next, we turn our attention to the financial market. Various attempts have been made to establish a micro-founded connection between the financial market and the real economy.[5] To ensure clarity and tractability, we adopt a reduced-form approach. We model the dynamics of the financial market using the standard mean-variance investor model and then impose a mechanism that links the evolution of the financial market to the real economy through expectations. This mechanism, known as the *financial accelerator*, is reflected in the IS curve in a way that is qualitatively similar to other studies.[6]

The financial market consists of risk-free and risky assets. Between periods t and $t+1$, the risk-free asset (such as a savings account) offers a nominal return equal to the nominal interest rate i_t, which is known at the moment of the investment de-

[5] Lengnick and Wohltmann (2016) derive the IS curve by assuming that agents' utilities depend on their holdings of financial assets. De Grauwe and Macchiarelli (2015) enhance the IS curve by the credit spread which evolves based on the lending market conditions.

[6] Given that the NK model does not incorporate capital, one can interpret the risky asset as an investment index opportunity. Consumers utilize this market for long-term savings, the credit conditions of firms depend on the state of this market. Consequently, the dynamics of the financial market affect the real side of the economy through credit channels, as in, for example, Bernanke and Gertler (1995), or through wealth effects, as modeled in Westerhoff (2012), or possibly both.

cision.[7] The infinitely lived risky asset has price p_t and pays an uncertain dividend y_t. For the sake of simplicity, we assume that $\{y_t\}_{t>0}$ are independently and identically distributed random variables with mean $\bar{y}$. The price of the risky asset evolves according to an asset pricing model with heterogeneous beliefs, see Campbell et al. (1997) and Brock and Hommes (1998). The dynamics derived in Appendix A are given by

$$p_t = \frac{1}{1+i_t}\left(\bar{E}_t p_{t+1} + \bar{y}\right) + \epsilon_t^p \,, \tag{3}$$

where $\bar{E}_t p_{t+1}$ stands for the average of investors' point expectations of the next period price of the asset, and ϵ_t^p is the stock price shock. It will be convenient to introduce the fundamental price of the risky asset as $p^f = \bar{y}/\bar{i}$, where $\bar{i} = \bar{r} + \pi^T$ is the nominal interest rate implied by the targeted real interest rate $\bar{r}$ and targeted inflation π^T.[8] Then we rewrite the dynamics above in terms of the *relative price deviation* defined by $q_t = (p_t - p^f)/p^f$, as

$$q_t = \frac{1}{1+i_t}\left(\bar{E}_t q_{t+1} + \bar{r} + \pi^T - i_t\right) + \epsilon_t^q = \frac{1}{1+i_t}\left(\bar{E}_t q_{t+1} + \bar{i} - i_t\right) + \epsilon_t^q \,, \tag{4}$$

where ϵ_t^q is the price shock in terms of relative deviation. This shock is assumed to be white noise independent of other shocks.

2.3 Expectations

We develop our model within the framework of heterogeneous expectations. Specifically, we introduce heterogeneity in expectations for our key variables, the output gap, inflation, and the price of the risky asset. This heterogeneity arises because while some agents trust the central bank targets (denoted as x^T for the output gap and π^T for inflation) and anticipate that the financial market will align with its fundamental level ($q = 0$), others may have doubts regarding the central bank's ability to achieve its targets or the accuracy of the asset pricing. Consequently, these agents adopt op-

[7] It implies the interest rate can be taken out of the expectations operator. This is a standard assumption in macroeconomics, see, e.g., Bernanke and Gertler (1999).

[8] As shown in Appendix A, the forward solution of the no-arbitrage equation depends on the expectations of future interest rates. We set these to be equal to the targeted interest rate to derive the fundamental price. The fundamental price is thus unrelated to the current interest rate whose deviation from the targeted level is considered to be temporary.

timistic or pessimistic predictions for these variables. Furthermore, we assume that when some predictions are proven right, all agents will learn about this and adjust their expectations in alignment with the more accurate predictions.

Our fundamental assumption is that the state of the financial market affects agents' expectations of the log-linearized output gap. Specifically, a bullish market, on average, makes agents more optimistic about tomorrow's output gap, while a bearish market makes them more pessimistic. This assumption can be explained by at least two possible mechanisms. First, agents may believe that better financial market conditions lead to higher wealth and consequently, higher future consumption. Second, firms may benefit from better financial conditions, having easier access to credit, which increases investment. Conversely, a bearish market can adversely affect their balance sheet and force firms to scale down production. In line with the literature, see, e.g., Bernanke et al. (1996), we refer to these mechanisms collectively as the *financial accelerator*.

We capture these ideas by employing the so-called large type limit introduced in Brock et al. (2005). Let us start with inflation expectations. We assume that agents can be divided into types according to the bias of their forecasts, where type h has bias b_h. Agents switch between these simple rules (different biased expectations) based on past square prediction errors. The heuristics are updated according to the discrete choice model with multinomial logit probabilities (see Manski and McFadden, 1981), and so the fractions of inflation bias h type at time t is given by

$$n^{\pi}_{t,h} = \frac{1}{Z_{t-1}} \exp[\gamma U^{\pi}_{t-1,h}] \,, \tag{5}$$

where $Z_{t-1} = \sum_{h=1}^{H} \exp[\gamma U^{\pi}_{t-1,h}]$ is the normalization constant, H is the total number of types, $U^{\pi}_{t-1,h}$ is the last period observed squared prediction error of inflation made by type h, and $\gamma \geq 0$ is the intensity of choice parameter. The higher the intensity of choice is, the more sensitive agents become with respect to the relative performance of predictions. Aggregate inflation expectations are given by a weighted average of the predictions of all types

$$\bar{E}_t \pi_{t+1} = \pi^T + \sum_{h=1}^{H} b_h n^{\pi}_{t,h} = \pi^T + \frac{1}{Z_{t-1}} \sum_{h=1}^{H} b_h \exp\left[-\gamma \left(\pi_{t-1} - b_h - \pi^T\right)^2\right] . \tag{6}$$

For any finite H, this equation can be written as

$$\bar{E}_t \pi_{t+1} = \pi^T + \frac{\frac{1}{H}\sum_{h=1}^{H} b_h \exp\left[-\gamma\left(\pi_{t-1} - b_h - \pi^T\right)^2\right]}{\frac{1}{H}\sum_{h=1}^{H} \exp\left[-\gamma\left(\pi_{t-1} - b_h - \pi^T\right)^2\right]} . \tag{7}$$

In Brock et al. (2005), H goes to infinity and the sample means in this equation are replaced with population means to derive the large type limit (LTL) dynamics. It is shown that important properties of the system, such as bifurcation scenarios, are preserved in the LTL approximation. Following Hommes and Lustenhouwer (2019b), we assume that the inflation expectation biases b_h are normally distributed around zero with a variance of s^2. The LTL is then given by:

$$\bar{E}_t \pi_{t+1} = \pi^T + \frac{\int b \exp\left[-\gamma\left(\pi_{t-1} - b - \pi^T\right)^2\right] \exp\left[-b^2/(2s^2)\right] \mathrm{d}b}{\int \exp\left[-\gamma\left(\pi_{t-1} - b - \pi^T\right)^2\right] \exp\left[-b^2/(2s^2)\right] \mathrm{d}b} . \tag{8}$$

The probability of agents choosing inflation expectations far from the target depends on the variance of the distribution of biases, s^2. If s^2 is small, most agents will have expectations close to the central bank's target, despite any recent deviations from it. If s^2 is large, the spread of possible biases is large, which means that any deviation of inflation from the target will cause many agents to update their expectations in the same direction. Parameter s^2 can thus be interpreted as an *inverse measure of anchoring* of expectations, see Hommes and Lustenhouwer (2019b).

A similar approach is adopted to model the output gap predictions. That is, agents are classified into types based on their forecasting biases, and the fractions of the types change based on past squared prediction errors. Importantly, we introduce the *financial accelerator* by assuming that the prediction biases for the output gap follow a normal distribution with a mean of λq_{t-1}, where the parameter $\lambda \geq 0$ measures the *strength of the financial accelerator*. If $\lambda = 0$, agents' biases are zero on average and the output gap expectations are distributed around the target as in Hommes and Lustenhouwer (2019b). But when $\lambda > 0$, expectations about the output gap deviate from its target in the direction suggested by the financial market. We set the variance of the output gap distribution to s^2, consistently with our assumptions about inflation expectation biases, and apply the LTL to obtain

$$\bar{E}_t x_{t+1} = x^T + \frac{\int b \exp\left[-\gamma\left(x_{t-1} - b - x^T\right)^2\right] \exp\left[-(b - \lambda q_{t-1})^2/(2s^2)\right] \mathrm{d}b}{\int \exp\left[-\gamma\left(x_{t-1} - b - x^T\right)^2\right] \exp\left[-\left(b - \lambda q_{t-1}\right)^2/(2s^2)\right] \mathrm{d}b} . \tag{9}$$

The financial market expectations are described using the same approach, with biases defined with respect to the relative price deviation $q^f \equiv 0$, corresponding to the fundamental price of the risky asset, p^f. When biases are normally distributed around 0 with variance s^2, the LTL reads

$$\bar{E}_t q_{t+1} = \frac{\int b \exp\left[-\gamma\left(q_{t-1} - b\right)^2\right] \exp\left[-b^2/(2s^2)\right] \mathrm{d}b}{\int \exp\left[-\gamma\left(q_{t-1} - b\right)^2\right] \exp\left[-b^2/(2s^2)\right] \mathrm{d}b} . \tag{10}$$

Using standard calculations (see, e.g., Appendix C in Hommes and Lustenhouwer, 2019b), the LTL equations for inflation, output gap, and asset prices are reduced to:

$$\begin{aligned} \bar{E}_t \pi_{t+1} &= A\pi^T + (1 - A)\pi_{t-1} \\ \bar{E}_t x_{t+1} &= A(x^T + \lambda q_{t-1}) + (1 - A)x_{t-1} , \\ \bar{E}_t q_{t+1} &= Aq^f + (1 - A)q_{t-1} \equiv (1 - A)q_{t-1} . \end{aligned} \tag{11}$$

Here, we introduced the *anchoring index* $A \in (0, 1]$ defined as

$$A = \frac{1}{1 + 2\gamma s^2} . \tag{12}$$

This index quantifies the strength of expectations being anchored to the targets. For a positive intensity of choice γ, as the variance of biases s changes from $+\infty$ to 0, the index A increases from 0 (indicating no anchoring) to 1 (reflecting full anchoring).[9] In equation (11), the expectations for each variable form a convex combination of the central bank's target (or zero deviation from the fundamental value) and the past value. Additionally, the financial accelerator influences the output gap target.

[9]When $\gamma = 0$, agents are equally likely to choose between all expectations, irrespective of past realizations. In this case, average expectations coincide with the mean of the distribution, and $A = 1$.

2.4 Monetary policy rule

To close the model, we assume that the nominal interest rate is set by a central bank according to the forward-looking Taylor type of interest rate rule:

$$i_t = \max\left\{\bar{r} + \pi^T + \phi_1\left(\bar{E}_t\pi_{t+1} - \pi^T\right) + \phi_2\left(\bar{E}_t x_{t+1} - x^T\right) + \Phi_3\left(\bar{E}_t q_{t+1}\right),\, 0\right\}, \quad (13)$$

where ϕ_1 and ϕ_2 are the reaction parameters to deviations of expectations about inflation and the output gap from their target values, and function $\Phi_3(\cdot)$ models a reaction to expectations about the relative asset price level. The inclusion of this function in the interest rate rule reflects a proposal for the central bank to directly respond to financial market mispricing.[10] Such *leaning-against-the-wind* (LAW) policy implies that the central bank increases its interest rate in response to positive bubbles (when $\bar{E}_t q_{t+1} > 0$) and decreases it in response to negative bubbles (when $\bar{E}_t q_{t+1} < 0$).

We specify function Φ_3 as follows

$$\Phi_3\left(\bar{E}_t q_{t+1}\right) = \begin{cases} \phi_3 \bar{E}_t q_{t+1} & \text{if } |\bar{E}_t q_{t+1}| > \alpha \\ 0 & \text{if } -\alpha \leq \bar{E}_t q_{t+1} \leq \alpha\,. \end{cases} \quad (14)$$

This piece-wise specification incorporates multiple scenarios. The parameter $\phi_3 > 0$ represents the strength of the central bank's response to market expectations regarding asset price deviations. However, the central bank only responds to deviations that exceed a certain threshold, denoted as α. When $\alpha = +\infty$, the central bank does not react to the financial market at all, and the Taylor rule in (13) becomes a standard rule, as examined in other studies. In the case where α is a positive but finite value, the central bank implements the LAW policy, but only after a sufficiently large bubble has formed. Lastly, when $\alpha = 0$, we are in a pure LAW policy regime.

Combining the dynamic equations of the real and financial sectors, (1) and (4), the LTL expectations (11), and the policy rule (13), we obtain the New Keynesian model with the financial market and heterogeneous expectations managed via the

[10]Note that rule (13) reacts to *relative* deviations of asset prices. This is consistent with its first two terms where the reaction is on log-linearized inflation and output gap.

Taylor rule. Its dynamics are described by the system

$$
\begin{aligned}
x_t &= A(x^T + \lambda q_{t-1}) + (1-A)x_{t-1} - \frac{1}{\sigma}\left(i_t - \bar{r} - A\pi^T - (1-A)\pi_{t-1}\right) + \epsilon_t^d \\
\pi_t &= \beta\left(A\pi^T + (1-A)\pi_{t-1}\right) + \kappa x_t + \epsilon_t^s \\
q_t &= \frac{1}{1+i_t}\left((1-A)q_{t-1} + \bar{r} + \pi^T - i_t\right) + \epsilon_t^q \\
i_t &= \max\Big\{\bar{r} + \pi^T + \phi_1(1-A)\left(\pi_{t-1} - \pi^T\right) + \phi_2(1-A)\left(x_{t-1} - x^T\right) + \\
&\qquad + \phi_2 A\lambda q_{t-1} + \Phi_3\left((1-A)q_{t-1}\right), 0\Big\}.
\end{aligned}
\tag{15}
$$

As in Hommes and Lustenhouwer (2019b), we set $x^T = \pi^T(1-\beta)/\kappa$, so that the output gap target is consistent with the inflation target. By plugging the interest rate into the other equations, the model can be written as a three-dimensional system in variables x, π, and q, buffeted by shocks. See technical Appendix C where the system is written explicitly. Note that we impose the zero lower bound (ZLB) in rule (13) and use this bound in all simulations of system (15) presented in the paper.[11]

The real sector, specifically the output gap, is impacted by the financial market via the financial accelerator, whose strength λ appears in the first equation. There is a further indirect impact if the interest rate rule conditions on the output gap, i.e., $\phi_2 > 0$. Moreover, if the central bank uses the LAW policy and conditions the interest rate on the state of the financial market via Φ_3, the real sector will be additionally affected via the interest rate. This last channel makes the LAW policy costly and controversial. The financial market in our model is not impacted by the real sector directly, but its dynamics are affected by the interest rate. Thus, it is monetary policy that channels the real sector's fluctuations to the financial market.

In the next section, we analyze the deterministic skeleton of the model, which is obtained by setting all shocks to zero. In Section 4, we study the fully-fledged stochastic system.

[11] A large literature including theoretical models of Arifovic et al. (2018); Hommes and Lustenhouwer (2019a,b) and experimental studies of Arifovic and Petersen (2017) and Hommes et al. (2019) analyze the policies at the ZLB. In our model, when the ZLB is imposed ($i_t \equiv 0$), the system has a unique equilibrium which is unstable for all parameterizations we use in this paper. (Formally, its stability condition coincides with condition (16) in Proposition 3.1, but for our parameterizations that condition is violated.) As the equilibrium is unstable and the ZLB is not our major concern, we do not discuss the properties of the ZLB equilibrium in the paper. However, we impose the bound on the interest rate in all our simulations.

3 Financial Accelerator and Monetary Policy

To gain an understanding of the properties of the New Keynesian model with financial accelerator (NK-FA model, henceforth), as described by (15), we will set all three shocks of this system, ϵ_t^d, ϵ_t^s, and ϵ_t^q, to zero and analyze the resulting nonlinear dynamical system. Appendix C contains derivations of the results presented below.

We start by clarifying why the monetary policy is needed.

Proposition 3.1. *Consider the case of no active monetary policy, $\phi_1 = \phi_2 = \phi_3 = 0$, so that the central bank pegs the interest rate in* (13) *by using $i_t = \bar{r} + \pi^T$ for all t. Then, NK-FA model* (15) *has the unique equilibrium $(x^T, \pi^T, 0)$. The equilibrium is globally stable if and only if*

$$A > A^* := \frac{\sqrt{\kappa^2 + 2(\beta+1)\kappa\sigma + (1-\beta)^2\sigma^2} - \kappa - \sigma(1-\beta)}{2\beta\sigma}, \tag{16}$$

where $A^* \in (0,1)$. If $A < A^*$, the dynamics are unbounded.

The equilibrium in terms of variables x, π, and q, identified in this proposition,

$$E^T = \left(x^T, \pi^T, 0\right) \ ,$$

is called the *targeted equilibrium*. In this equilibrium, the output gap and inflation are at their target levels, and the financial market price is at its fundamental level.

Proposition 3.1 demonstrates that under an interest rate peg, the system will converge to the targeted equilibrium from any starting point, provided that expectations are sufficiently anchored. However, if anchoring is weak, causing the index A defined in (12) to fall below the threshold A^* as defined in (16), the targeted equilibrium becomes unstable. Thus, strong anchoring of expectations is crucial for the success of the interest rate pegging policy. However, it is plausible that public expectations exhibit a high variance of biases (represented by a high value of s) and/or a strong reaction to past prediction errors (indicated by a high intensity of choice parameter γ). In such scenarios, expectations may not be sufficiently anchored. An active monetary policy then becomes necessary to mitigate the destabilizing effects of unanchored expectations.

When monetary policy is inactive, the financial accelerator does not influence the stability of dynamics in the NK-FA model. The financial market is unaffected by the real sector, and its dynamics converge independently, with no relevance to the convergence properties of the real sector. The situation changes when monetary policy is active. In the next section, we explore the case of active monetary policy without the LAW policy. Subsequently, in Section 3.2, we examine the fully-fledged NK-FA model with the LAW policy.

3.1 Effect of Financial Accelerator

This section studies the impact of the financial accelerator on the NK model dynamics. We set $\alpha = \infty$, so that $\Phi_3 \equiv 0$ in the Taylor rule (14). This exclusion of the LAW policy corresponds to a scenario in which the monetary authority does not respond to changes in financial market dynamics. It turns out that the financial accelerator leads to the emergence of a new, *non-targeted equilibrium* denoted as E^{NT}, given by

$$E^{NT} = (x^*, \pi^*, q^*) \ . \tag{17}$$

In this equilibrium, the output gap, inflation, and price are determined as follows:[12]

$$\begin{aligned} x^* &= x^T - (1 - \beta(1 - A))\,(\sigma - \phi_2) C q^* \\ \pi^* &= \pi^T - \kappa(\sigma - \phi_2) C q^* \\ q^* &= \frac{\bar{r} + \pi^T + A}{C\big(\phi_2\,(1 - \beta(1 - A))\,\sigma + (\phi_1 \sigma - \phi_2)\,(1 - A)\kappa\big)} - 1\,, \end{aligned}$$

where an auxiliary constant is defined as:

$$C = \lambda \frac{A}{((1 - A)(\sigma - \phi_2) - \sigma)\,(1 - \beta(1 - A)) - (1 - A)\,(\phi_1 - 1)\,\kappa} \,. \tag{18}$$

[12]One important qualification is that the point E^{NT}, defined here, is the fixed point of the system without imposing the ZLB. It is not an equilibrium of the model if it belongs to the *ZLB region*, which is defined as the set of combinations of the state variables x, π, and q that result in $i_t = 0$ according to the Taylor rule in (13). However, if E^{NT} does not belong to the ZLB region and $q^* > -1$, implying that the asset price is positive and well-defined, E^{NT} is an equilibrium of the model. In all formal results, we specify that the non-targeted (and later LAW-) equilibrium exists whenever the point E^{NT} does not belong to the ZLB region.

The following result formalizes this finding:

Proposition 3.2. *Consider the NK-FA model with no LAW policy ($\alpha = \infty$). Then:*

1) For $\lambda = 0$, E^T is an equilibrium of the model. It is locally stable if and only if $\phi_1^d < \phi_1 < \phi_1^u$, where

$$\begin{aligned} \phi_1^d &:= 1 - \frac{1-\beta(1-A)}{\kappa}\left(\frac{A}{1-A}\sigma + \phi_2\right) \\ \phi_1^u &:= 1 + \frac{1+\beta(1-A)}{\kappa}\left(\frac{2-A}{1-A}\sigma - \phi_2\right) . \end{aligned} \tag{19}$$

If one of the inequalities is reversed, the dynamics become unbounded.

2) For $\lambda > 0$, in addition to an equilibrium E^T, the model has the non-targeted equilibrium, E^{NT}, defined in (17), *if it does not belong to the ZLB region. Only one of the two equilibria can be stable, with E^{NT} gaining stability when E^T loses it through a transcritical bifurcation of eigenvalue $+1$.*

When $\lambda = 0$ (no financial accelerator) and $\alpha = \infty$ (no monetary response to financial market dynamics), the real sector decouples from the financial market, mirroring the Hommes and Lustenhouwer (2019b) model. Proposition 3.2 *1)* states that stability conditions for our targeted equilibrium are the same as in that model.[13]

Proposition 3.2 *2)* reveals that with the financial accelerator channel, the NK-FA model can have an additional equilibrium E^{NT}. Fig. 1 illustrates the stability regions of both targeted and non-targeted equilibria in coordinates (ϕ_1, ϕ_2) for four different values of the financial accelerator strength λ. Other parameters are set to[14]

$$\kappa = 0.024\,,\ \beta = 0.99\,,\ \sigma = 0.157\,,\ A \approx 0.051\,. \tag{20}$$

The stability region of the targeted equilibrium is shown in gray. Labels on its borders indicate the type of bifurcation: flip, transcritical (TR), or Neimark-Sacker (NS).[15] Note that the left border of the stability region of E^T reflects the standard

[13]It is easy to verify that bounds in (19) coincide with those in Hommes and Lustenhouwer (2019b) Proposition 1. Note also, that for $\phi_1 = \phi_2 = 0$, local stability conditions in Proposition 3.2, when rewritten in terms of A, match global stability condition (16) in Proposition 3.1.

[14]All values are from Woodford (1999), except for the anchoring index A which we set using the parameterization from Hommes and Lustenhouwer (2019b). With the intensity of choice $\gamma = 60,000$ (based on Cornea-Madeira et al. 2019) and the variance $s^2 = (5/400)^2$, (12) yields $A = 4/79 \approx 0.051$.

[15]We place labels in the same positions across different panels for easy comparison of the regions.

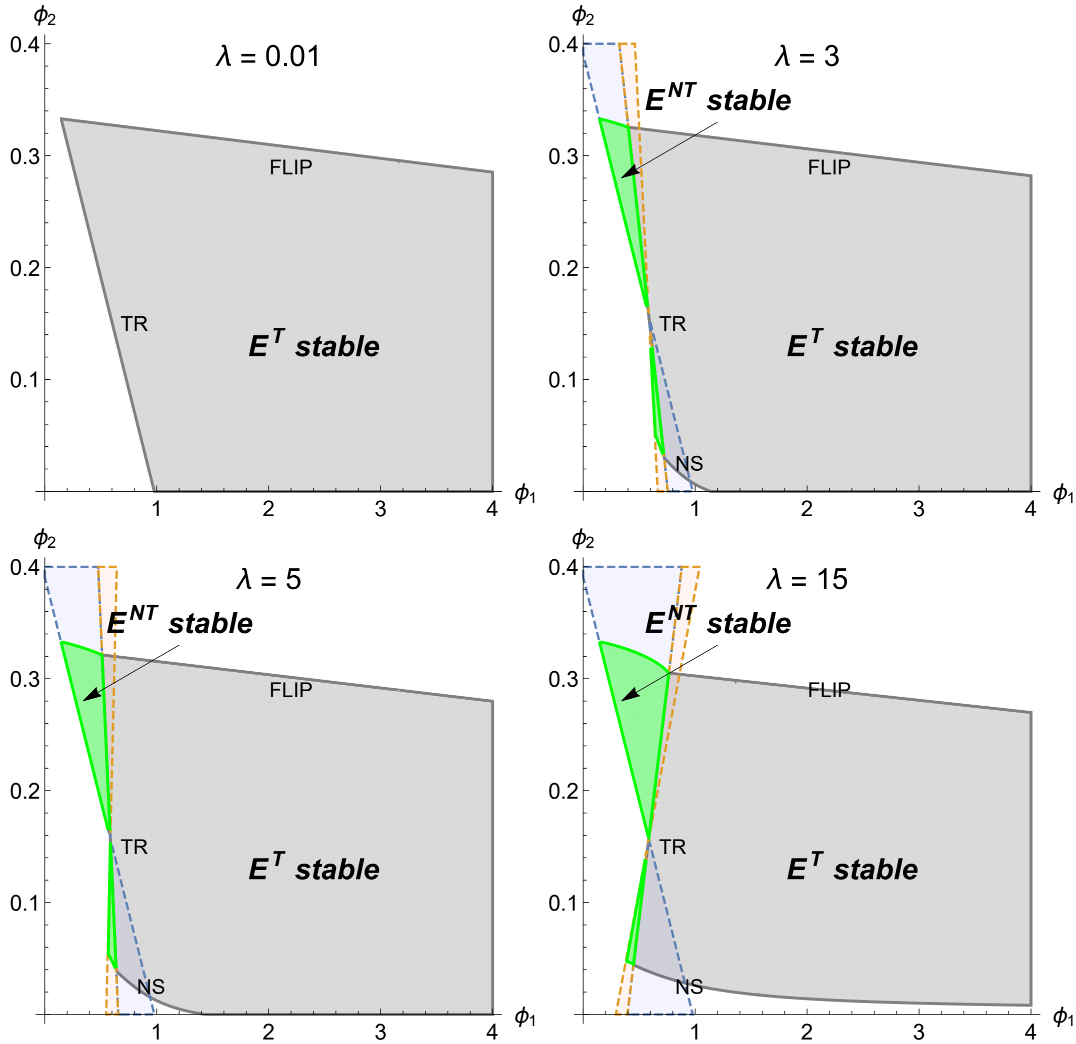


Figure 1: Stability of the NK-FA model under no LAW policy ($\alpha = \infty$) for four λ values in the (ϕ_1, ϕ_2) plane. The stability region of the targeted equilibrium E^T is shaded in gray, while the stability region of the non-targeted equilibrium E^{NT} is in green. Regions where E^{NT} exists are in light-blue ($q^* < 0$) and light-red ($q^* > 0$). *Top left*: $\lambda = 0.01$. *Top right*: $\lambda = 3$. *Bottom left*: $\lambda = 5$. *Bottom right*: $\lambda = 15$. Remaining parameters are as defined in (20).

Taylor principle that ϕ_1 should be high enough for stability. In the non-targeted equilibrium, when it exists outside the ZLB region, the financial market may be undervalued ($q^* < 0$) or overvalued ($q^* > 0$). The blue and red regions indicate these cases, respectively, while the green region signifies stability for E^{NT}. According to Proposition 3.2, E^{NT} is nonexistent when the financial accelerator λ is equal to 0. The top-left panel of Fig. 1 illustrates its absence even for a positive but small value,

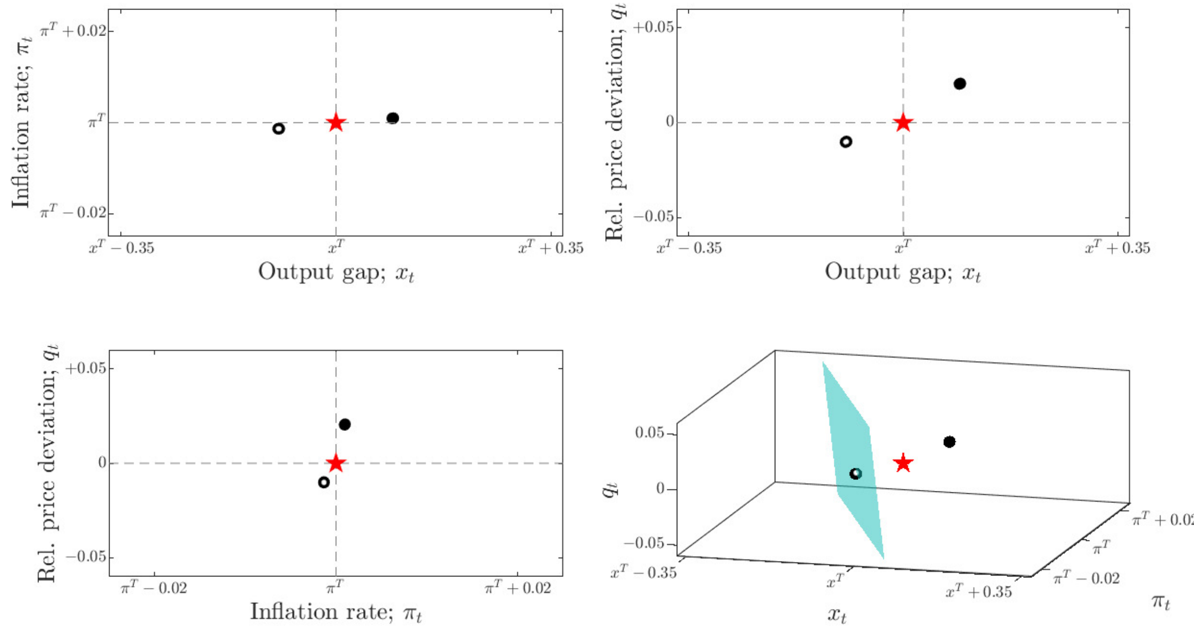


Figure 2: Attractor of the NK-FA model immediately after a flip bifurcation. The targeted equilibrium E^T, marked by a star, is unstable, while the stable 2-period cycle is indicated by dots, with the empty dot located inside the ZLB region. The border of the ZLB region, separating the empty dot and the star, is highlighted in cyan in the phase space (bottom-right panel). The parameters are $\phi_1 = 2$, $\phi_2 = 0.30475$, $\lambda = 0.01$, and others as in (20).

$\lambda = 0.01$.

As the financial accelerator strength, λ, increases, the shape of the stability region for E^T changes, as seen when comparing the top-left panel with other panels. Furthermore, the region where the non-targeted equilibrium E^{NT} is stable emerges through a *transcritical* bifurcation (eigenvalue $+1$) of E^T occurring when ϕ_1 decreases. The stability region of the non-targeted equilibrium expands with increasing λ.

The targeted equilibrium E^T can also lose stability through two other bifurcation types. A bifurcation of eigenvalue -1 occurs by increasing ϕ_2. Numerical simulations confirm that this is a *degenerate flip* bifurcation.[16] The presence of the ZLB in the

[16] At the bifurcation point, the eigenvector associated with the eigenvalue -1 spans a segment filled with 2-period cycles. A point of non-differentiability of the model, which represents a period point of one of these 2-period cycles, is located at the border of the ZLB region. The related 2-period

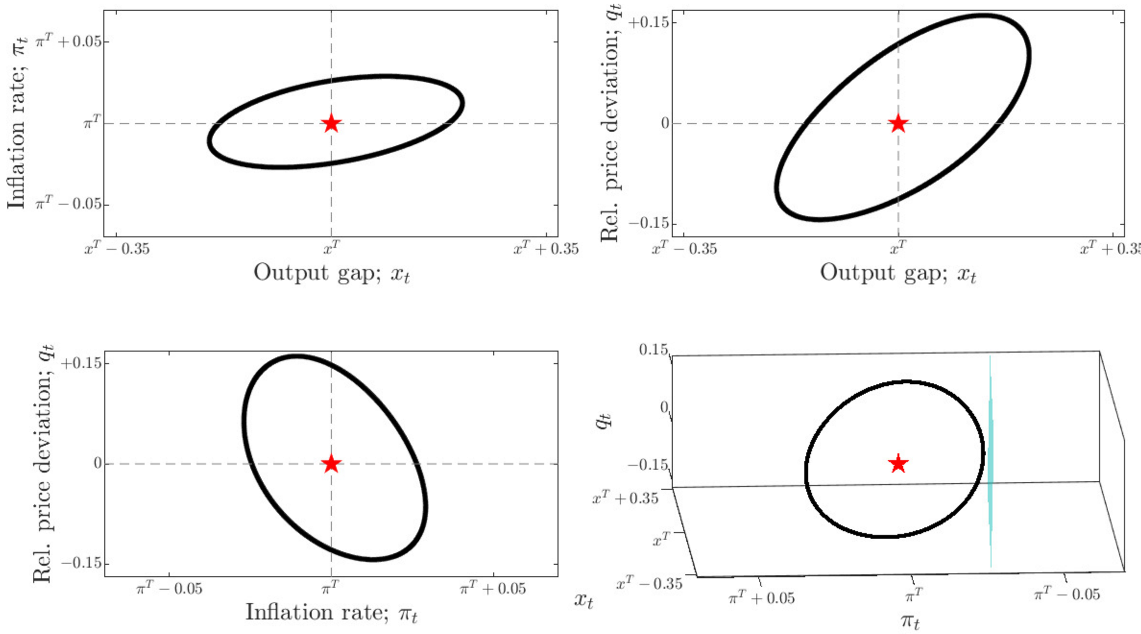


Figure 3: Attractor of the NK-FA model after a Neimark-Sacker bifurcation. The targeted equilibrium E^T, marked by a star, is unstable, while the limit cycle around it is stable. The parameters are $\phi_1 = 1.05$, $\phi_2 = 0.01002$, $\lambda = 5$, and others as in (20).

interest rate rule limits oscillations after bifurcation. Instead of unlimited oscillations, 2-period cycle dynamics emerge, as illustrated in Fig. 2. The bottom right panel shows all three state variables, while the other panels are projections. The red star represents E^T, unstable for the chosen parameters (see the caption). Dynamics converge to the stable 2-period cycle indicated by two black dots. One of these dots (an empty one) belongs to the ZLB region. The hyperplane bordering the ZLB in the 3D panel separates this empty dot and the star.

Another possible bifurcation scenario occurs for sufficiently high λ (see all panels of Fig. 1 except for the first one). A *Neimark-Sacker* bifurcation (two complex conjugate eigenvalues have a modulus equal to 1) occurs by reducing ϕ_1 when ϕ_2 is close to zero. After this bifurcation, an attracting closed orbit persists, as illustrated in Fig. 3. Furthermore, it is observed that with a further reduction of ϕ_1, the stable closed

cycle undergoes a persistent border collision, as described in detail in Sushko and Gardini (2010). As a result, after the bifurcation, only this 2-period cycle persists and remains stable.

orbit comes into contact with the ZLB region and eventually disappears.

The bifurcations of the targeted equilibrium in the NK-FA model are further illustrated in Fig. 4, where 12 bifurcation diagrams with respect to the policy parameter ϕ_2 are shown (with fixed $\phi_1 = 1.05$). These diagrams are organized in four columns that correspond to different parameterizations and show three panels each with the dynamics of π_t (top), x_t (middle), and q_t (bottom). The first and second columns correspond to the case with no LAW policy, discussed here. The third and fourth columns of panels will be discussed in the next section.

The left column of Fig. 4 shows the bifurcation diagrams of the NK model with no financial accelerator ($\lambda = 0$). For small values of ϕ_2, the targeted equilibrium is stable. However, increasing ϕ_2 leads to instability of E^T due to a degenerate flip bifurcation. The 2-period cycle, illustrated earlier in Fig. 2, can be observed as a pair of distinctive points for x and rather close points for π and q when $\phi_2 \approx 0.321$. Soon after the bifurcation, the 2-period cycle bifurcates further to a more complicated attractor. For large values of ϕ_2, the attractors exhibit persistent distortions. Specifically, the average values of the output gap and inflation rate are below their targets, while the financial market is consistently overvalued (bullish). The dashed black line in each panel illustrates it by showing the average values of output, inflation, and relative price deviation for every value of the bifurcation parameter ϕ_2. The distortions are influenced by the ZLB constraint. When the output gap and/or inflation rate are low, the monetary policy suggests lowering the interest rate. However, the ZLB becomes binding, weakening the effectiveness of the monetary policy. In contrast, there is no constraint when the interest rate needs to be increased. Moreover, when the output gap and inflation rate are consistently lower than their targeted values, the interest rate will also be below the long-run target. This generates a persistent bullish financial market even without the financial accelerator channel. All these factors contribute to the asymmetry observed in the attractors, which are periodic or chaotic and differ from the targeted equilibrium.

When the model is enhanced by the financial accelerator channel, as shown in the second column panels of Fig. 4, there are notable differences. First, when ϕ_2 is close to zero, the targeted equilibrium is now unstable, surrounded by an attracting closed orbit, illustrated earlier in Fig. 3. The vertical segment around zero deviations

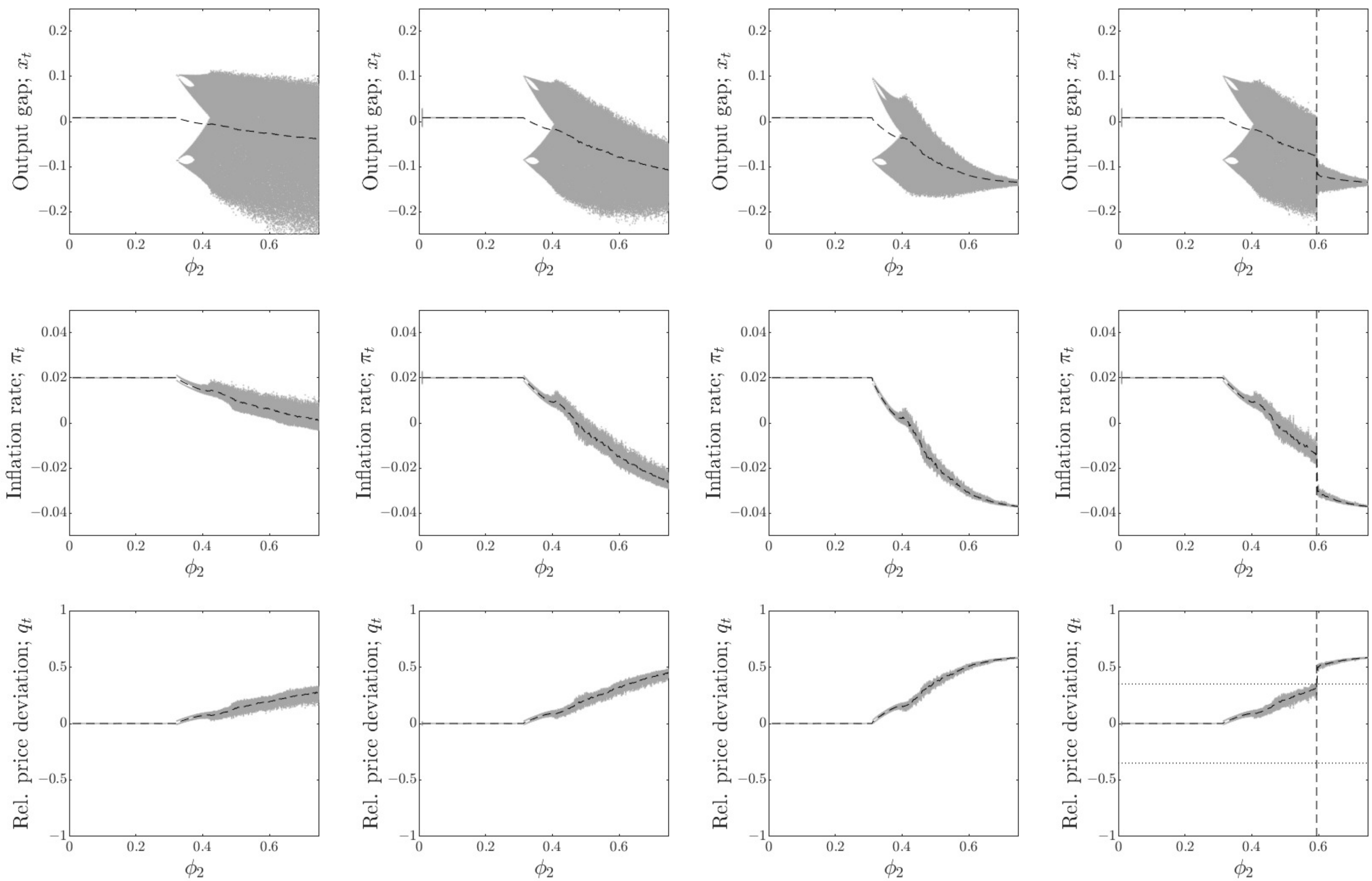


Figure 4: Bifurcation diagrams of three state variables of the NK-FA model, with respect to $\phi_2 \in [0, 0.8]$. *1st column*: $\lambda = 0$, $\alpha = \infty$ (no financial accelerator, no LAW policy). *2nd column*: $\lambda = 5$, $\alpha = \infty$ (no LAW policy). *3rd column*: $\lambda = 5$, $\phi_3 = 0.03$ and $\alpha = 0$ (unconditional LAW policy). *4th column*: $\lambda = 5$, $\phi_3 = 0.03$ and $\alpha = 0.35$ (LAW policy). In all cases, $\phi_1 = 1.05$ and the remaining parameters are as in (20).

can be seen for all three variables. Second, for high values of ϕ_2, the distortions of the variables from their targeted levels become larger. These additional distortions occur because optimistic expectations on the output gap are influenced by the bullish financial market. To counteract these optimistic expectations, the monetary policy keeps the nominal interest rate (and also the real interest rate since $\phi_1 > 1$ in this numerical example) higher than it would be based solely on the realized output gap. This justifies the output gap and the inflation rate deviating (on average) even further from their targets compared to when the financial accelerator channel is absent. Additionally, this leads to a lower interest rate than it would be without the financial accelerator channel, contributing to the maintenance of a bullish financial market. Therefore, the financial accelerator generates a distorting effect on expectations which, through the monetary policy channel, can contribute to the fueling of a financial bubble.

In summary, the financial accelerator channel reduces the stability region of the targeted equilibrium, may lead to the emergence of the non-targeted equilibrium, and generates additional optimism contributing to further distortions of inflation and output gap, thereby fueling a bullish financial market when instability occurs. All these effects are negative. However, there is a silver lining. The amplitudes of fluctuations along complicated attractors for high ϕ_2 decrease, as evidenced in Fig. 4.

3.2 LAW Policy

We now introduce the leaning against the wind (LAW) policy in our NK-FA model. Recall that by (14), the LAW policy becomes active after a certain threshold $\alpha \geq 0$ of mispricing in the financial market is reached. The monetary rule reacts to these deviations of the financial market with strength $\phi_3 > 0$.

It turns out that the LAW policy generates an additional *LAW-equilibrium* denoted as E^{LAW}, given by:

$$E^{LAW}(\phi_3) = (x^*(\phi_3), \pi^*(\phi_3), q^*(\phi_3)) \ . \tag{21}$$

In this equilibrium, the output gap, inflation, and price are determined as

$$x^*(\phi_3) = x^T - (1-\beta(1-A))\left(\sigma - \phi_2 - \phi_3 \frac{1-A}{\lambda A}\right) C q^*$$

$$\pi^*(\phi_3) = \pi^T - \kappa\left(\sigma - \phi_2 - \phi_3 \frac{1-A}{\lambda A}\right) C q^*$$

$$q^*(\phi_3) = \frac{\bar{r} + \pi^T + A}{C\left(\phi_2(1-\beta(1-A))\sigma + (\phi_1\sigma - \phi_2)(1-A)\kappa\right) + \phi_3 \frac{1-A}{\lambda A}\left(\sigma A\left(1-\beta(1-A)\right) - (1-A)\kappa\right)} - 1$$

where an auxiliary constant C is defined in (18). Note that this equilibrium converges to the non-targeted equilibrium E^{NT} when $\phi_3 \to 0$.

In the special case of $\alpha = 0$, we have the *unconditional LAW policy*, that is, the monetary rule reacts to *any* deviation of the financial market.[17] Here, we investigate the case of a LAW policy applied when mispricing reaches a strictly positive threshold. While such a policy cannot affect the stability of the targeted equilibrium (as it is never applied in the neighborhood of E^T, where $q^* = 0$), it turns out that it reduces the volatility of fluctuations on the periodic and complex attractors and also decreases the region of instability of the system. We summarize the formal result as follows:

Proposition 3.3. *Consider the NK model with a financial accelerator and a conditional LAW policy with a threshold $\alpha > 0$ and a strength of $\phi_3 > 0$. Then:*

1) *There are three possible equilibria: the targeted equilibrium E^T, the non-targeted equilibrium E^{NT} as defined in (17), if it does not enter the $|q| > \alpha$ and ZLB regions, and the LAW equilibrium E^{LAW} as defined in (21), if it does not enter the $|q| < \alpha$ and ZLB regions.*

2) *The stability conditions of the targeted equilibrium E^T are the same as described in Proposition 3.2.*

3) *Both stable equilibria, E^T and E^{LAW}, can coexist. The parameter ϕ_3 affects the stability of E^{LAW} (but not of the other equilibria): the higher the value of ϕ_3, the larger the stability region of E^{LAW}.*

Proposition 3.3 reveals that the system can have two coexisting stable equilibria when a LAW policy is implemented. This is a novel feature of the model. The co-existence of

[17] This case is analyzed in Appendix B.

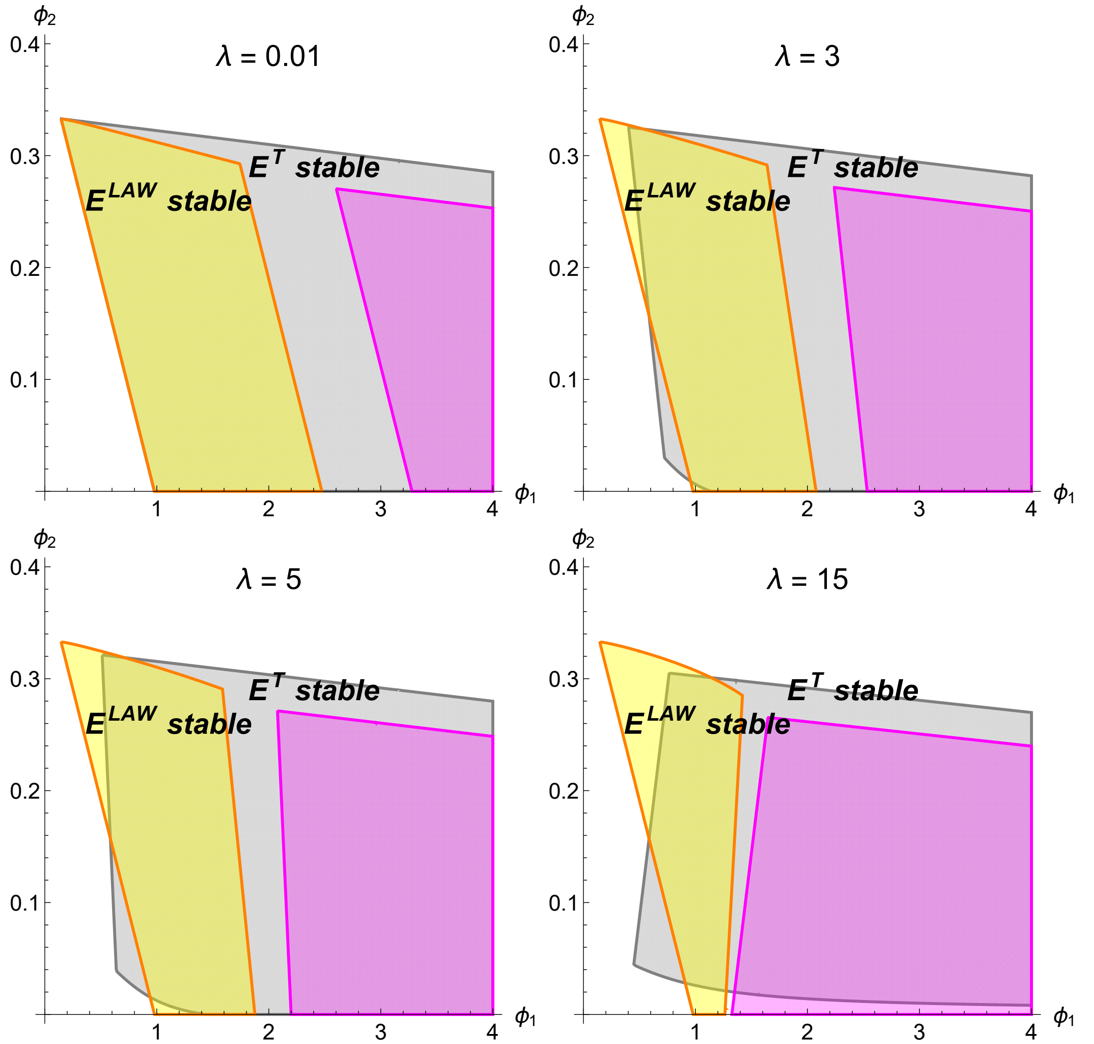


Figure 5: Stability of the NK-FA model under the LAW policy with $\alpha = 0.35$ and $\phi_3 = 0.2$ for four λ values in the (ϕ_1, ϕ_2) plane. The stability region of the targeted equilibrium E^T is shaded in gray, while the stability region of the LAW equilibrium E^{LAW} is in yellow. The stability region of E^T under unconditional LAW policy ($\alpha = 0$) is in magenta. *Top left*: $\lambda = 0.01$. *Top right*: $\lambda = 3$. *Bottom left*: $\lambda = 5$. *Bottom right*: $\lambda = 15$. Remaining parameters are as defined in (20).

stable equilibria is illustrated in Fig. 5, analogous to Fig. 1. For the same four different values of the financial accelerator strength λ, we reproduce the stability regions of E^T in gray in coordinates (ϕ_1, ϕ_2).[18] Given that $\phi_3 > 0$, the non-targeted equilibrium is replaced by the LAW equilibrium defined in (21) (for $\phi_3 = 0.2$ in this case). The

[18]As mentioned earlier and stated in Proposition 3.3 *2)*, the local stability of E^T is not affected. This is in stark contrast to the unconditional LAW case where $\alpha = 0$; see also Fig. 12 in Appendix B.

yellow region indicates parameters when E^{LAW} is stable. This region is larger than the stability region of E^{NT}, but it is important to note that in the LAW equilibrium, mispricing is generally larger. The overlap of the gray and yellow regions indicates the coexistence of the targeted and LAW equilibria. Additionally, we observe a portion of the stability region of the LAW equilibrium that does not overlap with the targeted equilibrium. Simulations show that this portion increases as, given the threshold α, we increase the strength ϕ_3. In other words, a more restrictive LAW policy enlarges the stability region of the non-desirable equilibrium, allowing for a broader range of parameter values where the model converges to some equilibrium. Finally, in Fig. 5, the magenta region replicates the stability region of E^T under the unconditional LAW policy (i.e., when $\alpha = 0$) from Fig. 12 in Appendix B. From the comparison of these regions, we conclude that a less reactive application of the LAW policy (i.e., applied with a positive threshold, $\alpha = 0.35$ in this case) significantly enlarges the stability region of the targeted equilibrium compared to the unconditional LAW policy.

Overall, through the comparison of stability regions, we observe the positive effect of the LAW policy when applied not overly reactively but after the financial market deviation reaches a threshold α. This policy effectively mitigates destabilizing effects caused by the financial accelerator channel. Upon comparing the four panels in Fig. 5, it becomes evident that the significance of this positive effect increases with the strength of the financial accelerator.

To study the effect of the LAW policy after bifurcations, we revisit the bifurcation diagrams in Fig. 4, where the targeted equilibrium loses stability through a degenerate flip bifurcation for high ϕ_2 values. Notable differences are observed in the third and fourth columns, representing the unconditional LAW policy ($\alpha = 0$) and the LAW policy with a threshold of $\alpha = 0.35$, respectively. The third column indicates that the unconditional LAW policy decreases the amplitude of fluctuations arising when the targeted equilibrium becomes unstable. However, it also leads to increased average disturbances in the output gap, inflation, and the financial market price (as indicated by the black dashed curve). On the other hand, the fourth column illustrates the scenario where the LAW policy is activated only for sufficiently high price deviations. This implies that immediately after the bifurcation, the policy is not applied, and fluctuations mirror those depicted in the second column, where no LAW policy is present, consequently leading to lower average disturbances compared

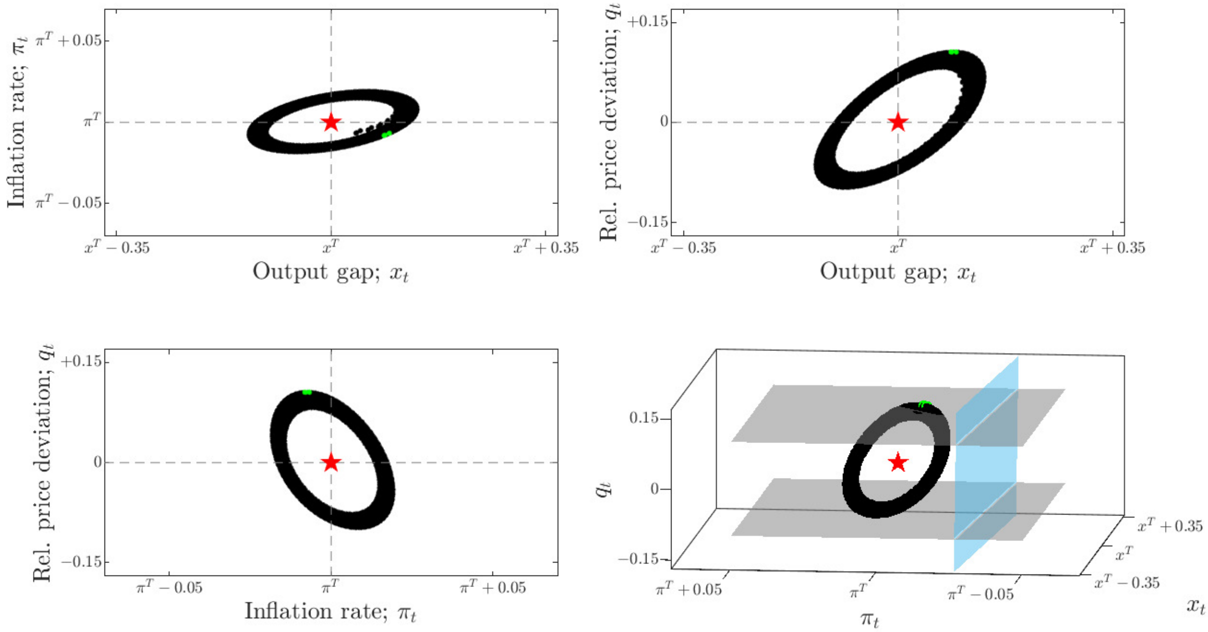


Figure 6: Attractor of the NK-FA model with conditional LAW policy after a Neimark-Sacker bifurcation. The points of the complex attractor that are in the region where the LAW policy applies are in green. The border of the ZLB region is a vertical azure plane, while the region where the LAW policy does not apply is in between two horizontal planes. The parameters are as in Fig. 3 except that $\phi_3 = 0.1$ and $\alpha = 0.1$.

to the unconditional LAW case. However, for values of ϕ_2 that are sufficiently high, the LAW policy comes into effect, and the dynamics now resemble those in the third column, resulting in lower fluctuations but higher average disturbances.

By examining the region below the Neimark-Sacker bifurcation curve in the parameter space (ϕ_1, ϕ_2) depicted in Fig. 5, the numerical simulation shown in Fig. 6 demonstrates that the introduction of a conditional LAW policy dampens the fluctuations generated by the NK model.[19] This damping effect of the conditional LAW policy is beneficial and conceals a bifurcation that gives rise to a chaotic attractor. Specifically, when the stable closed orbit resulting from a Neimark-Sacker bifurcation of the targeted equilibrium intersects with the region where the conditional LAW pol-

[19] In the case of the unconditional LAW policy, convergence to the targeted equilibrium is achieved. However, as we have seen, this policy destabilizes the targeted equilibrium in other parts of the parameter space.

icy is implemented, the attractor persists but may exhibit chaotic behavior. This phenomenon can be understood as a border-collision bifurcation, as discussed in Sushko and Gardini (2010) and Avrutin et al. (2019). The boundary of the region where the conditional LAW policy is active approaches the targeted equilibrium as the parameter α is reduced. Therefore, the amplitude of the fluctuations can be controlled by adjusting the value of the policy parameter α.

In summary, a LAW policy has several effects on the New Keynesian model augmented by the financial accelerator. On the positive side, it expands the region of the parameter space where the model converges to some equilibrium and reduces the amplitude of fluctuations when they occur, contributing to lower volatility. On the other hand, it may converge to a non-desired equilibrium and a reduction in the amplitude of fluctuations may come at the cost of increased average mispricing. Importantly, activating the LAW policy not blindly but conditionally on the financial market mispricing reaching a certain threshold allows the policymaker to achieve a middle ground between having no policy and implementing an unconditional LAW policy.

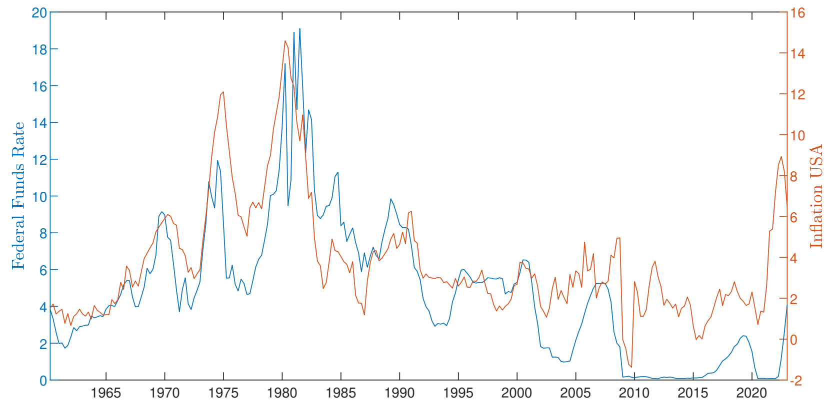
4 Stochastic Simulations and Monetary Policy

We have established that the targeted equilibrium maintains local stability within the NK-FA model, even when beliefs are dispersed and unanchored, as long as the central bank adopts an appropriate monetary policy. While there are important considerations regarding the multiplicity of equilibria under the LAW policy and the consequences of crossing the stability region, the existence of a substantial parameter space supporting the stability of the targeted equilibrium (see, e.g., Fig. 5) suggests that the monetary authority can effectively manage unanchoring, even in the presence of the financial accelerator channel. However, the local stability of equilibrium does not imply an absence of a dynamical response of the system to shocks. Indeed, Section 3 only studied the deterministic version of the model, switching off all noise terms in system (15). In this section, we delve into the analysis of time series generated by the *stochastic* NK-FA model to gain a deeper understanding of the effects of monetary policies. Additionally, we qualitatively compare the simulated time series with empirical data on inflation, output gap, interest rates, and financial markets for the United States, whose evolution is illustrated in Fig. 7.

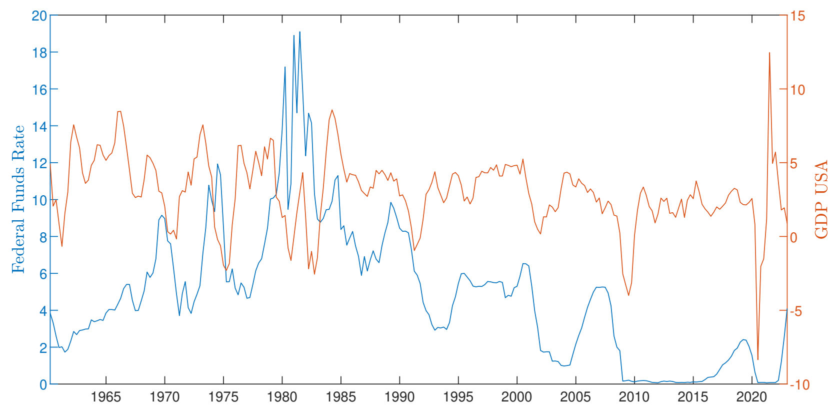
As described in Clarida et al. (2000), the monetary policy experience in the United States can be summarized in two fundamental phases: the pre-Volcker period (until the 1970s) and the Volcker period (after the 1970s). During the pre-Volcker period, an accommodating monetary policy rule was in place, which was reflected in the estimation of $\phi_1 < 1$ in the monetary rule (13). Under this policy, an increase in the inflation rate was followed by a *reduction* in the real interest rate. This self-fulfilling mechanism contributed to the propagation and persistence of high inflation rates observed during that period (refer to Fig. 7). Specifically, for $\phi_1 < 1$, the nominal interest rate increased but did not fully compensate for the rise in inflation, resulting in a lower real interest rate. The consequences of this accommodating policy rule were prolonged periods of high inflation accompanied by a decrease in the real interest rate. As a result, the output gap (GDP) increased, further fueling inflation in subsequent periods. On the other hand, the Volcker period was characterized by a non-accommodating, restrictive monetary policy rule, as indicated by the estimated value of $\phi_1 > 1$ in (13). In this phase, an increase in the inflation rate was met with increases in both nominal and real interest rates. High inflation rates were associated with, or followed by, output gaps lower than the long-run target, while low inflation rates were associated with, or followed by, output gaps higher than the long-run target.

In the next two sections, we will analyze the dynamics of the NK-FA model under the accommodating monetary policy rule and the restrictive monetary policy rule scenarios, respectively. To incorporate stochastic shocks into the model, we introduce the IID Gaussian processes ϵ_t^s, ϵ_t^d, and ϵ_t^q in system (15). These processes have zero mean and standard deviations of 0.1%, 0.1%, and 10%, respectively. As discussed above, our primary focus will be on examining the stochastic dynamics in the case when the targeted equilibrium is stable, which will guide our choice of the policy parameters ϕ_1, ϕ_2, ϕ_3 and α. Similarly, we set $\lambda = 0.01$ implying a rather low strength of the financial accelerator channel. The remaining parameters are maintained at fixed values, as specified in (20), that is, $\kappa = 0.024$, $\beta = 0.99$, and $\sigma = 0.157$.

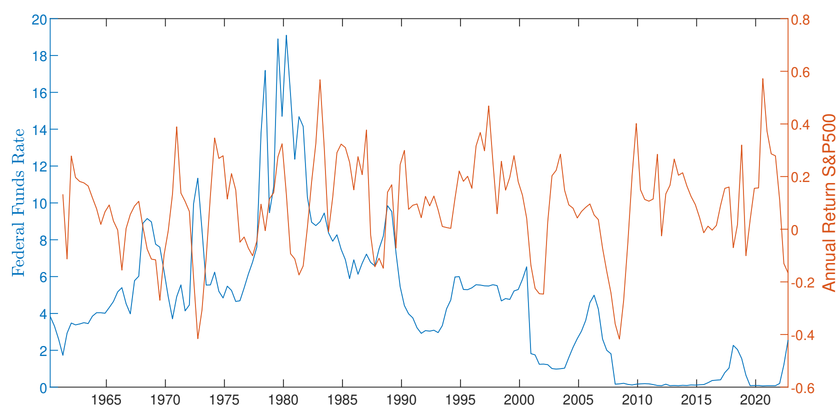
Finally, to underscore the impact of anchoring, we conduct two sets of simulations.



(a) Consumer Price Index (percentage rate of change of CPI, year on year)



(b) Gross Domestic Product (percentage rate of change of GDP, year on year)



(c) S&P500 relative annual return

Figure 7: Empirical data for the United States, source *Refinitiv*. Each panel shows the evolution of the Fed Funds Effective Rate (left axis) versus one of the statistics related to the state variables of the model (right axis).

In the "unanchored" case, we set $A = 0.051$, as in (20). For the second, "anchored" case, we use a larger value of $A \approx 0.571$.[20]

4.1 Accommodating monetary policy for controlling inflation

The investigation of the deterministic skeleton of the New Keynesian model reveals that under an accommodating monetary policy rule (i.e., $\phi_1 < 1$), the stability of the targeted equilibrium E^T is lost when either ϕ_1 decreases (then the non-targeted equilibrium emerges) or ϕ_2 increases (then a 2-period cycle emerges), see Fig. 1. In such a scenario, introducing a LAW policy only results in creating an additional LAW equilibrium, which may be stable when the targeted equilibrium is not. Compare the green region in Fig. 1 with the yellow region in Fig. 5.

In our simulations, we use the following parameter values: $\lambda = 0.01$, $\phi_1 = 0.95$, $\phi_2 = 0.1$, and $\phi_3 = 0.001$. Regarding monetary policy, our main focus is on comparing the case without a LAW policy (i.e., $\alpha = \infty$), with the case with a LAW policy applied after the threshold of $\alpha = 0.35$. The deterministic skeleton of the model exhibits local stability for the targeted equilibrium under these parameter settings.

The time series in Fig. 8 illustrate the simulated dynamics. In these and other simulations, the blue line represents the dynamics of the targeted equilibrium. The upper panels show the scenario with unanchored expectations (low A), while the lower panels show the scenario with anchored expectations (high A). Each panel compares the dynamics without a LAW policy (in red), and with a LAW policy activated at the threshold $\alpha = 0.35$ (in black). In the anchored case, the dynamics closely follow the targeted equilibrium. In contrast, in the unanchored case, despite the equilibrium's stability, noise induces prolonged deviations from the equilibrium. The LAW policy appears to have a limited effect on stabilizing the financial market, with almost no visible differences between the black and red lines in the top-right panel. However, manipulating the interest rate based on the financial market's state has a more pronounced impact on the time series of the output gap. This observation suggests that the output gap is more sensitive to interest rate changes than the financial market.

[20] These two parameterizations and terminology are taken from Hommes and Lustenhouwer (2019b). For the "anchored" case, they set $\gamma = 60,000$ and $s^2 = (1/400)^2$, which means $A = 4/7 \approx 0.571$. See footnote 14.

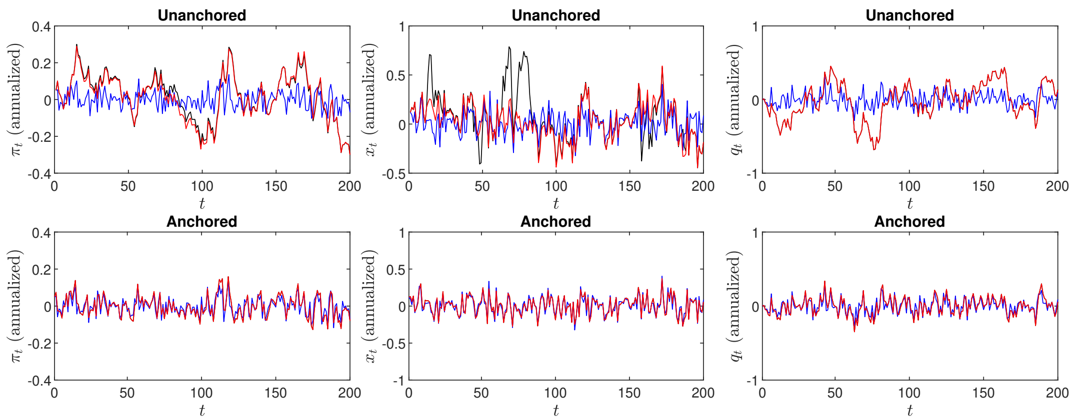


Figure 8: Time series of three state variables of the NK-FA model with $\lambda = 0.01$. *Top panels*: unanchored expectations. *Bottom panels*: anchored expectations. Blue: the targeted equilibrium. Red: no LAW policy. Black: LAW policy with $\phi_3 = 0.001$, $\alpha = 0.35$. Parameters κ, β and σ are set as in (20). Policy parameters: $\phi_1 = 0.95$, $\phi_2 = 0.1$.

Consequently, the LAW policy exhibits side effects that were not apparent in the equilibrium analysis from Section 3. For example, when the financial market is undervalued, the LAW policy imposes a lower interest rate, significantly influencing the output gap dynamics and generating a wave of high output gap levels.[21]

We can quantify the effect of the policy by calculating the volatility of the simulated time series, as presented in Table 1. A comparison between the third and fourth columns reveals that the LAW policy doubles the volatility of the output gap. Furthermore, in additional simulations, we consider an unconditional LAW policy (i.e., $\alpha = 0$), as shown in the last column. The effect is even more pronounced when an unconditional LAW policy is implemented.

Abstracting from the effects of the LAW policy, it is worth noting that in this particular monetary policy setting, the financial market in our model exhibits countercyclical dynamics. Periods characterized by deflation, low output gaps, and low interest rates can give rise to a financial bubble. However, this bubble eventually bursts when inflation rates rise, leading to an increase in interest rates. This pattern aligns with empirical observations from the pre-Volcker period, as depicted in Fig. 7.

[21] We have examined the robustness of these findings to larger values of λ, and the same mechanisms and insights were observed. The same holds for the simulations reported later in Fig. 9.

		no FA, no LAW	FA, no LAW	FA, cond LAW	FA, uncond LAW
		$\lambda = 0$ $\alpha = \infty$	$\lambda = 0.01$ $\alpha = \infty$	$\lambda = 0.01$ $\alpha = 0.35$ $\phi_3 = 0.001$	$\lambda = 0.01$ $\alpha = 0$ $\phi_3 = 0.001$
x_t	mean	x^T	x^T	x^T	x^T
	volatility	0.18%	0.18%	0.39%	0.58%
	skewness	−0.0032	−0.0127	−0.6482	−0.2289
π_t	mean	π^T	π^T	π^T	π^T
	volatility	0.32%	0.32%	0.36%	0.42%
	skewness	−0.0034	−0.0337	−0.0501	−0.0570
q_t	mean	0.1%	0.18%	0.30%	0.40%
	volatility	26.40%	26.82%	26.42%	26.22%
	skewness	0.1488	0.0215	0.3790	0.3237

Table 1: Mean, volatility, and skewness of the simulated dynamics of the NK-FA model with unanchored expectations. Statistics were carried out on a sample of $1,000$ time series of 200 periods each. Parameters are in the second row. The other parameters are $\phi_1 = 0.95$, $\phi_2 = 0.1$, and as in (20).

To illustrate the impact of bifurcations on instability studied in Section 3, we conduct a second set of simulations. In this scenario, we increase the sensitivity to the output gap ($\phi_2 = 0.34$) and to the financial market ($\phi_3 = 0.02$), while keeping other parameters unchanged. Under these parameters and unanchored expectations, the deterministic skeleton of the model exhibits cyclical, or chaotic, dynamics for all three policy scenarios considered. Introducing shocks, we obtain simulations as shown in Fig. 9. The mean, volatility, and skewness over $1,000$ of such simulations are reported in Table 2.

In the simulations presented in the upper panels, we observe that the implementation of a LAW policy does not reduce the volatility of the financial market. Instead, it has the unintended consequence of increasing the volatility of both the inflation rate and the output gap. A comparison of the statistics in Table 2 helps quantify these effects. Specifically, the LAW policy results in a doubling of the overpricing in the financial market (from 3.3% to over 6%) and a doubling of the volatility of the inflation rate (from 0.32% to 0.59%). Furthermore, in this configuration, the output gap, on average, remains negative and decreases further when the LAW policy is implemented. The inflation rate, which is already negative at approximately −0.2%,

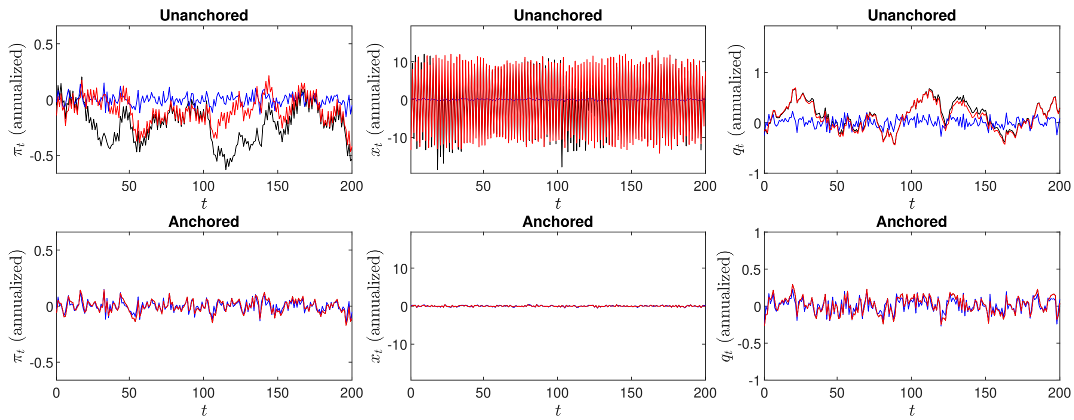


Figure 9: Time series of three state variables of the NK-FA model with $\lambda = 0.01$. *Top panels*: unanchored expectations. *Bottom panels*: anchored expectations. Blue: the targeted equilibrium. Red: no LAW policy. Black: LAW policy with $\phi_3 = 0.02$, $\alpha = 0.35$. Parameters κ, β and σ are set as in (20). Policy parameters: $\phi_1 = 0.95$, $\phi_2 = 0.34$.

decreases to around -0.4% when the LAW policy is in place. It is worth noting that deflation is primarily caused by the ZLB, which binds every second period. However, the implementation of the LAW policy exacerbates these phenomena, especially when it is employed unconditionally (see the last column). As before, the side effects of the LAW policy vanish when expectations are anchored, as observed in the case shown in the lower panels of Fig. 9. In this scenario, market fluctuations are limited, and the LAW policy with a threshold of $\alpha = 0.35$ is rarely activated.

Overall, we have observed that the implementation of a LAW policy alongside an accommodating monetary policy leads to several side effects. Interestingly, the financial market plays a role in mitigating hyperinflation when expectations of the output gap are strongly anchored to the dynamics of the financial market (indicated by a large value of λ).[22] In this scenario, despite the presence of an accommodating monetary policy (where $\phi_1 < 1$, as seen in the pre-Volcker era), inflation is effectively controlled by the financial market. When inflation increases, the nominal interest rate rises, which dampens the financial market. As a result, the output gap, which would otherwise expand under the accommodating monetary policy, decreases. Consequently, inflation decreases due to the contraction of the output gap. A similar,

[22]While we find these results interesting, we do not report them here due to space constraints. Instead, we offer a qualitative explanation of the underlying mechanism.

		no FA, no LAW	FA, no LAW	FA, cond LAW	FA, uncond LAW
		$\lambda = 0$ $\alpha = +\infty$	$\lambda = 0.01$ $\alpha = +\infty$	$\lambda = 0.01$ $\alpha = 0.35$ $\phi_3 = 0.02$	$\lambda = 0.01$ $\alpha = 0$ $\phi_3 = 0.02$
x_t	mean	-0.005	-0.005	-0.010	-0.010
	volatility	9.58%	9.60%	9.19%	9.09%
	skewness	-0.0022	-0.0022	0.1028	0.1063
π_t	mean	-0.002	-0.002	-0.004	-0.004
	volatility	0.32%	0.32%	0.59%	0.59%
	skewness	-0.0054	-0.0619	-0.0996	-0.0361
q_t	mean	3.3%	3.3%	6.40%	7.25%
	volatility	26.60%	26.27%	28.71%	28.64%
	skewness	-0.0000	0.0218	0.0838	0.0818

Table 2: Mean, volatility, and skewness of the simulated dynamics of the NK-FA model with unanchored expectations. Statistics were carried out on a sample of $1,000$ time series of 200 periods each. Parameters are in the second row. The other parameters are $\phi_1 = 0.95$, $\phi_2 = 0.34$, and as in (20).

yet opposite, mechanism operates in the case of low inflation. In such situations, financial bubbles and crashes help regulate inflation. Therefore, in these cases, the monetary authority should refrain from intervening in the financial market to stabilize it, as doing so would be counterproductive and could prolong periods of high or low inflation.

4.2 Restrictive monetary policy for controlling inflation

We now turn to the case when the LAW policy is combined with a restrictive monetary policy ($\phi_1 > 1$). We set the parameters to $\lambda = 0.01$, $\phi_1 = 2$, $\phi_2 = 0.2$, and $\phi_3 = 0.01$. Under this parameter configuration, considering both levels of A (corresponding to unanchored and anchored expectations) and the remaining parameters as defined in (20), the deterministic skeleton of the model exhibits a locally stable targeted equilibrium under three different monetary policy scenarios: no LAW policy, LAW policy applied after the threshold $\alpha = 0.35$, and unconditional LAW policy (that is threshold $\alpha = 0$). When random shocks are introduced into the model, the dynamics are illustrated in Fig. 10, where the targeted equilibrium is shown in blue,

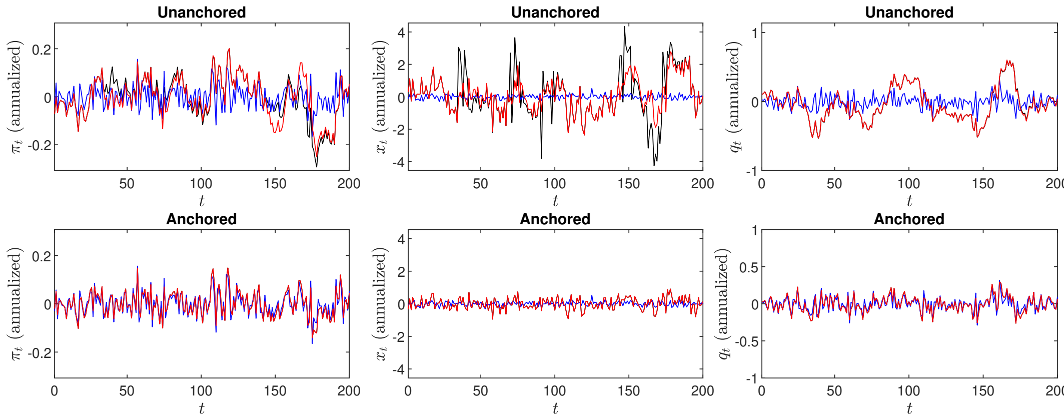


Figure 10: Time series of three state variables of the NK-FA model with $\lambda = 0.01$. *Top panels*: unanchored expectations. *Bottom panels*: anchored expectations. Blue: the targeted equilibrium. Red: no LAW policy. Black: LAW policy with $\phi_3 = 0.01$, $\alpha = 0.35$. Parameters κ, β and σ are set as in (20). Policy parameters: $\phi_1 = 2$, $\phi_2 = 0.2$.

the simulations without the LAW policy are in red, and the simulations with the LAW policy are in black. We again focus on the unanchored case, as in the anchored case the noise is not amplified.

The simulations reveal a negative correlation between the output gap and inflation. Specifically, a sequence of positive inflation shocks can trigger a sustained upward trend in inflation due to unanchored expectations. In response, the monetary policy reacts by raising the interest rate. The combined effect of high inflation and high interest rates leads to a higher real interest rate, as the monetary policy becomes more aggressive in countering inflation (i.e., $\phi_1 > 1$). Consequently, the increased interest rates offset the impact of high inflation, resulting in a negative effect on the output gap. This scenario is characterized by waves of stagflation, featuring high inflation, high interest rates, and a low output gap, as depicted in the upper panels of Fig. 10, which are reversed in periods of a high output gap and low inflation rates due to an active non-accommodating monetary policy. This suggests that a non-accommodating monetary policy aimed at combating inflation is particularly useful for addressing stagflation in the case of unanchored expectations.

Regarding the financial accelerator channel, we observe that under a restrictive monetary policy that aggressively combats inflation, the real sector aligns with the financial market. Furthermore, the output gap and the financial market exhibit a

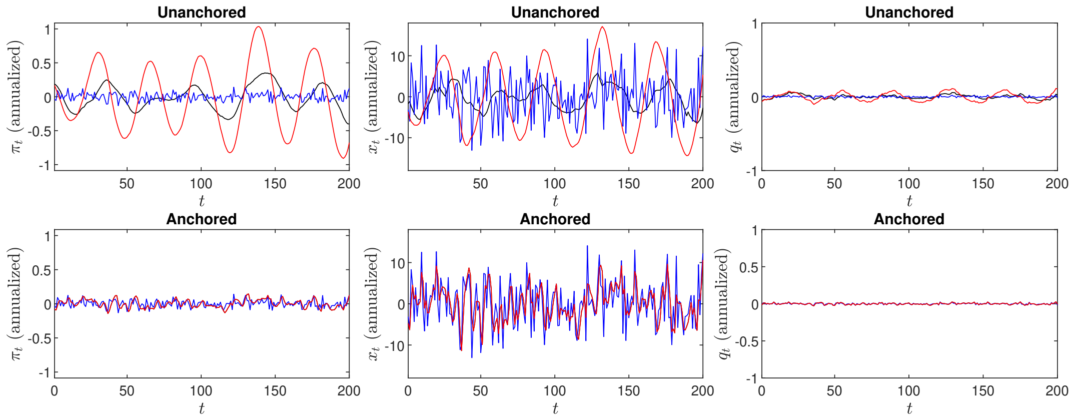


Figure 11: Time series of three state variables of the NK-FA model with $\lambda = 5$. *Top panels*: unanchored expectations. *Bottom panels*: anchored expectations. Blue: the targeted equilibrium. Red: no LAW policy. Black: LAW policy with $\phi_3 = 0.05$, $\alpha = 0.05$. Parameters κ, β and σ are set as in (20). Policy parameters: $\phi_1 = 1.05$, $\phi_2 = 0.011$.

negative correlation with inflation. A period of high inflation, triggered by a sequence of positive inflation shocks and amplified by unanchored expectations, prompts a monetary policy response characterized by high nominal interest rates. These elevated interest rates counteract inflation, initiating a recessionary dynamic in the output gap (which is further amplified by unanchored expectations). Simultaneously, the financial market experiences a crash, initially triggered by the rise in interest rates and amplified by unanchored expectations. This pattern appears consistent with empirical observations related to the Volcker period, as depicted in Fig. 7.

Regarding the LAW policy, to observe its effects, we compare the time series in the upper panels of Fig. 10, generated for no LAW policy (in red) and LAW policy with $\alpha = 0.35$ (in black). These time series demonstrate that a LAW policy with a parameter of $\phi_3 = 0.01$ generates larger fluctuations in the output gap and inflation rate. However, when examining the financial market, we observe no difference between the time series observed with or without LAW policy. The LAW policy does not produce these side effects when expectations are anchored, as seen in the lower panels of Fig. 10.

Up until now, the time series generated by the stochastic NK-FA model have mainly highlighted the negative effects associated with the LAW policy. However, by increasing the value of the financial accelerator parameter λ to 5, the NK-FA model becomes more sensitive to shocks. Thus, setting $\phi_1 = 1.05$ and $\phi_2 = 0.011$, we find that the deterministic skeleton of the model generates stable orbits through Neimark-Sacker bifurcation both without and with LAW policy. However, as suggested by the deterministic analysis, the LAW policy contributes to reducing the amplitude of fluctuations. This is confirmed by the time series generated by the model with shocks. Compare the time series in Fig. 11, generated with no LAW policy (in red) and with LAW policy and relatively low threshold $\alpha = 0.05$ (in black). In the upper panels, which illustrate the case of unanchored expectations, we can observe that the LAW policy helps reduce the amplitude of fluctuations. Conversely, no effect of the LAW policy is noted when expectations are anchored, as depicted in the lower panels of Fig. 11.[23]

5 Conclusion

The effectiveness of the Leaning Against the Wind policy as a monetary tool is an ongoing and active research area. Whether it should be implemented is unclear, with strong arguments against it based on cost-benefit analysis (Svensson, 2017). Policy-makers argue that its poor implementation may adversely affect the economic situation for an extended period. For instance, Bernanke (2000) includes "the apparent attempt to 'prick' the stock market bubble in 1989–91, which helped induce an asset-price crash" in the list of three most important monetary policy mistakes in Japan. However, there is no clear-cut answer, and the debates are ongoing.

In this paper, we address the efficiency of the LAW policy from the perspective of the literature on heterogeneous expectations. This literature emphasizes that expectations of economic agents are not rational but adapt to the previous forecasting errors. The degree of anchoring these expectations to targeted values then becomes crucial for conducting monetary policy. Our model extends the recent work by Hommes and Lustenhouwer (2019b), incorporating the financial sector and establishing its connection with the real sector through the financial accelerator channel. Additionally, the interest rate rule in our model allows the real sector to influence the financial market.

[23]The fluctuations generated after a Neimark-Saker bifurcation have a low amplitude. This is why, in the numerical simulations in Fig. 11, a lower value of threshold $\alpha = 0.05$ is used. A higher value of α (e.g., $\alpha = 0.35$) would require a higher variance of the noise to activate the LAW policy.

Our analysis reveals that when the financial accelerator channel is present, the system may converge to an equilibrium different from the one targeted by the monetary authority. Moreover, with the implementation of the LAW policy, two stable equilibria, the targeted one and an alternative equilibrium, can coexist. Consequently, the LAW policy may have a counterproductive effect as agents may coordinate their behavior around that alternative equilibrium. Furthermore, the stability region of the targeted equilibrium, within the space of policy parameters, diminishes with the LAW policy. Our stochastic simulations demonstrated that even if conditional (on financial market mispricing) implementation of the LAW policy mitigates the issue of diminishing the stability region of the targeted equilibrium, it does not reduce but can even increase the volatility of economic variables.

Even though we have also seen that the LAW equilibrium may reduce the magnitude of fluctuations along periodic or chaotic attractors in situations where stable equilibria are absent, the overall findings of this paper suggest that the LAW policy is not an efficient tool for achieving the targeted equilibrium. This contrasts somewhat with several other recent papers. As mentioned earlier, in experiments conducted by Bao and Zong (2019) and Hennequin and Hommes (2023) interest rate rules could effectively reduce financial market bubbles. The model presented in Schmitt and Westerhoff (2021) demonstrates that conditioning on price momentum, rather than on mispricing, may be an efficient way to implement the LAW policy. While it is important to note that these papers do not include the real sector, unlike our study, this suggests that further analytical and experimental studies in this area are needed.

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APPENDIX

A Financial Market

Traders in the financial market divide their financial wealth between risk-free and risky assets. The risk-free asset interest is i_t. Denote the number of shares purchased by trader j in period t as $z_{j,t}$. Each share costs p_t and yields a payoff $p_{t+1}+y_{t+1}$. The wealth of the trader in period $t+1$ is thus given by

$$W_{j,t+1} = (1+i_t)\left(W_{j,t} - p_t z_{j,t}\right) + \left(p_{t+1} + y_{t+1}\right) z_{j,t} \, .$$

Traders are myopic mean-variance optimizers, so the demand for shares $z_{j,t}$ is obtained by solving

$$\max_{z_{j,t}} \left\{ E_{j,t}\left[W_{j,t+1}\right] - \frac{a}{2} V_{j,t}\left[W_{j,t+1}\right] \right\} \, , \tag{A.1}$$

where $a/2$ measures the degree of risk aversion. We assume that traders differ in their beliefs about the next period's price and denote their point expectations of the price as $p^e_{j,t+1}$. We further assume that they correctly predict dividends, so that $E_{j,t}[y_{t+1}] = \bar{y}$, and that they have homogeneous expectations about the variance of total payoff $V[p_{t+1}+y_{t+1}]$. Solving (A.1), we obtain the risky asset demand:

$$z_{j,t}\left(p_t\right) = \alpha \left(p^e_{j,t+1} + \bar{y} - (1+i_t)\, p_t\right) \, ,$$

where $\alpha = 1/(aV[p_{t+1}+y_{t+1}])$. Assuming the zero outside supply of shares, the market equilibrium condition becomes

$$\sum_j z_{j,t}\left(p_t\right) = 0 \quad \Leftrightarrow \quad \sum_j \bar{E}_t p_{t+1} + \bar{y} - (1+i_t)\, p_t = 0 \, ,$$

where $\bar{E}_t p_{t+1}$ is the average of the traders' price expectations, $p^e_{j,t+1}$. The equilibrium price of the risky asset is, therefore, given by

$$p_t = \frac{1}{1+i_t}\left(\bar{E}_t p_{t+1} + \bar{y}\right) . \qquad \text{(A.2)}$$

Up to the shock term, this is the equation (3) in the main text.

To derive the *fundamental price* level, we assume homogeneous expectations and iterate (A.2) forward. We obtain (see also Hennequin and Hommes, 2023) that

$$p_t = \frac{1}{1+i_t}\left(E_t \lim_{K\to\infty}\left[p_{t+K}\prod_{k=1}^{K}\frac{1}{1+i_{t+k}}\right] + E_t\left[\bar{y}\sum_{s=1}^{\infty}\prod_{k=1}^{s}\frac{1}{1+i_{t+k}}\right]\right) . \qquad \text{(A.3)}$$

The solution depends on the expectations of future interest rates.

We derive the fundamental price as the time-consistent solution of (A.2) under homogeneous expectations and common knowledge of central bank targeting goals. This means the agents make computations above discounting the future with the interest rate $\bar{i}$, i.e., the equilibrium nominal interest rate. Given that the equilibrium real interest rate is pinned down by the model's parameters and the inflationary target is constant, agents derive the targeted nominal rate

$$\bar{i} = \bar{r} + \pi^T = \pi^T + \frac{1}{\beta} - 1 .$$

The solution of (A.3) in such case is well known to be $p^f = \bar{y}/\bar{i}$. This is what we use in the model as the benchmark price with respect to which agents may decide if the bubble is being developed.

Finally, with the variable $q_t = (p_t - p^f)/p^f$, from Eq. (A.2), it follows that

$$q_t = \frac{1}{1+i_t}\left(\bar{E}_t q_{t+1} + 1 + \frac{\bar{y}}{p^f}\right) - 1 = \frac{1}{1+i_t}\left(\bar{E}_t q_{t+1} + \bar{i} - i_t\right) .$$

Adding the shock, we obtain Eq. (4) in the main text.

B Unconditional LAW Policy

In this appendix, we examine the case of the so-called *unconditional LAW* policy, referring to the monetary rule that reacts to *any* deviation of the financial market. In other words, we set the threshold $\alpha = 0$ in the policy function (14). The strength of the reaction to price deviation is denoted by $\phi_3 > 0$.

We have the following result.

Proposition B.1. *Consider the NK-FA model under the unconditional LAW policy ($\alpha = 0$). Then:*

1) *There are at most two equilibria, the targeted equilibrium E^T, and the LAW equilibrium E^{LAW} defined in* (21)*, if it does not belong to the ZLB region. Only one of the two equilibria can be stable, with E^{LAW} gaining stability when E^T loses it through a transcritical bifurcation of eigenvalue* $+1$.

2) *For $\lambda < \dfrac{1}{\sigma}$ and $\phi_3 = A\left(\sigma - \phi_2\right)$, E^T is stable when the conditions in* (19) *are satisfied.*

Proposition B.1 indicates that the qualitative dynamics of the NK model do not change substantially when introducing the unconditional LAW policy. Even with the application of the policy, we still observe two coexisting equilibria. The LAW equilibrium replaces the non-targeted equilibrium and can gain stability through a transcritical bifurcation when the fundamental equilibrium loses stability. The other bifurcation scenarios are also similar as illustrated in Fig. 12.

Considering $\lambda > 0$ and the other parameters as defined in (20), an exploration of the parameter space (ϕ_1, ϕ_2) reveals that the unconditional LAW policy *reduces* the stability region of the targeted equilibrium. This can be observed by comparing the gray stability regions of E^T in the four panels of Fig. 1, where $\alpha = \infty$, with the gray stability regions in the four panels of Fig. 12, where $\alpha = 0$. In the latter case, we use

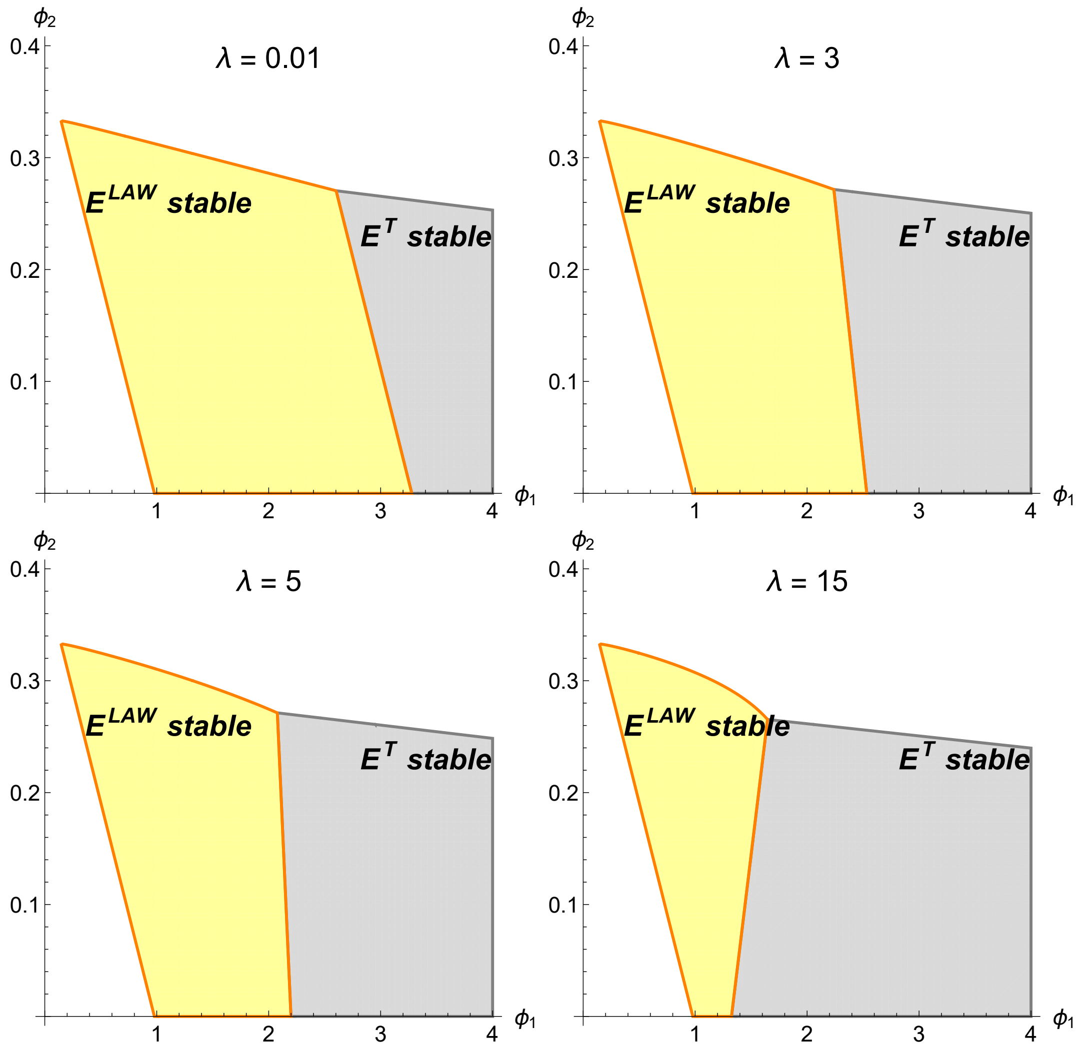

Figure 12: Stability of the NK-FA model under unconditional LAW policy ($\alpha = 0$) with $\phi_3 = 0.2$ for four λ values in the (ϕ_1, ϕ_2) plane. The stability region of the targeted equilibrium E^T is shaded in gray, while the stability region of the LAW equilibrium E^{LAW} is in yellow. *Top left*: $\lambda = 0.01$. *Top right*: $\lambda = 3$. *Bottom left*: $\lambda = 5$. *Bottom right*: $\lambda = 15$. Remaining parameters are as defined in (20).

a monetary policy strength of $\phi_3 = 0.2$. The effect is further illustrated by plotting the stability regions in the parameter space (ϕ_1, ϕ_2, ϕ_3) in Fig. 13. It can be seen that the LAW policy, especially when applied strongly would destabilize the targeted equilibrium.

Fig. 12 clearly illustrates that the shrinkage of the stability region of E^T is accompanied by the enlargement of the stability region of E^{LAW}, depicted in yellow. This

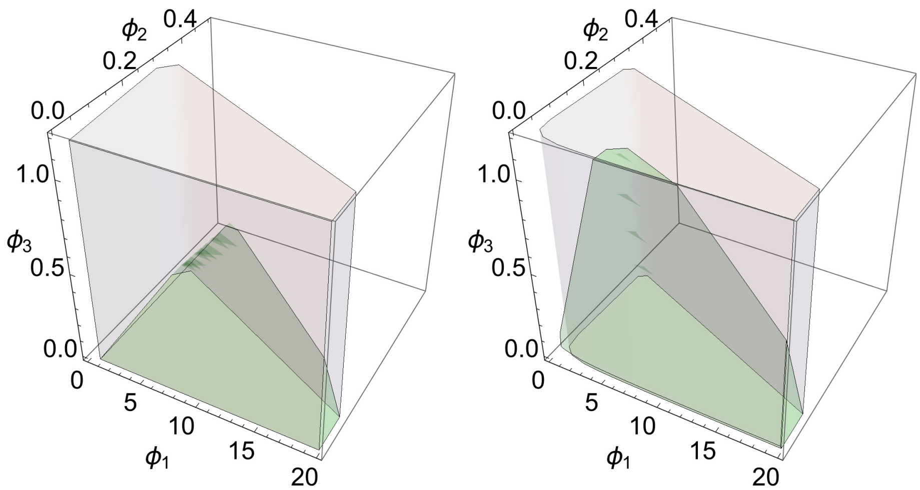

Figure 13: Comparison of regions of local stability for the targeted equilibrium E^T in the NK-FA model, without LAW policy ($\alpha = \infty$, gray set) and with the unconditional LAW policy ($\alpha = 0$, green set) in the (ϕ_1, ϕ_2, ϕ_3) space. *Left panel*: $\lambda = 0.01$. *Right panel*: $\lambda = 15$. Remaining parameters are as defined in (20).

region expands with the unconditional LAW policy, indicating that there are specific combinations of ϕ_1 and ϕ_2 values for which stability of the alternative equilibrium can only be achieved when the active LAW policy is implemented. This can be observed by comparing the green regions in the four panels of Fig. 1 with the yellow regions in the four panels of Fig. 12.

The negative effect of the unconditional LAW policy is also seen when the parameters cross the stability region, as we illustrated in Fig. 4 of the main text. Its 3rd column shows the bifurcation diagram under the unconditional LAW policy. We observe that the LAW policy decreases the amplitude of the periodic or chaotic fluctuations that emerge when the targeted equilibrium loses stability. However, it is important to note that the disturbances in the output gap, inflation, and the overvaluation of the financial market are not reduced, but rather further increased. Overall, the unconditional LAW policy fails to resolve the issues caused by the financial accelerator effect. If anything, it exacerbates the instability by shrinking the stability region of the targeted equilibrium and enlarging the stability region of the alternative equilibrium.

C Technical Appendix

The New Keynesian model with the financial market under heterogeneous expectations is derived in the paper as (15). We analyze here the deterministic skeleton of this system and set all three shocks to zero.

Recall, that the central bank targets the output gap to be consistent with the inflation target, that is $x^T = \pi^T(1-\beta)/\kappa$. Then, operating the change of variables $\tilde{x}_t = x_t - x^T$ and $\tilde{\pi}_t = \pi_t - \pi^T$, we can focus on the NK model in *deviations* from the targeted-equilibrium $E^T = \left(x^T, \pi^T, 0\right)$. We study the equilibria of the model and their stability by focusing first on the region where the zero lower bound (ZLB) is not binding, and later we address the region where the ZLB is binding.

ZLB is not binding. By plugging the interest rate rule into the equations for the output gap and asset prices, and employing the piece-wise linear function $\Phi_3(\cdot)$ in (14), we obtain two subcases. When $|q_t| < \alpha$, there is no interest rate reaction to the financial market, and the dynamical system becomes

$$
\begin{aligned}
\tilde{x}_t &= \left(1 - \frac{\phi_2}{\sigma}\right)(1-A)\,\tilde{x}_{t-1} + \frac{1-\phi_1}{\sigma}(1-A)\,\tilde{\pi}_{t-1} + \left(1-\frac{\phi_2}{\sigma}\right)A\lambda q_{t-1} \\
\tilde{\pi}_t &= \beta(1-A)\,\tilde{\pi}_{t-1} + \kappa\tilde{x}_t \\
q_t &= \frac{(1-A-\phi_2 A\lambda)\,q_{t-1} - \phi_1(1-A)\,\tilde{\pi}_{t-1} - \phi_2(1-A)\,\tilde{x}_{t-1}}{1+\bar{r}+\pi^T+\phi_1(1-A)\,\tilde{\pi}_{t-1}+\phi_2(1-A)\,\tilde{x}_{t-1}+\phi_2 A\lambda q_{t-1}}\,.
\end{aligned}
$$

Instead, when $|q_t| \geq \alpha$, the leaning against the wind (LAW) policy is in place, and the dynamical system becomes

$$
\begin{aligned}
\tilde{x}_t &= \left(1 - \frac{\phi_2}{\sigma}\right)(1-A)\,\tilde{x}_{t-1} + \frac{1-\phi_1}{\sigma}(1-A)\,\tilde{\pi}_{t-1} + \left(1-\frac{\phi_2}{\sigma}\right)A\lambda q_{t-1} - \frac{\phi_3}{\sigma}(1-A)\,q_{t-1} \\
\tilde{\pi}_t &= \beta(1-A)\,\tilde{\pi}_{t-1} + \kappa\tilde{x}_t \\
q_t &= \frac{((1-\phi_3)(1-A) - \phi_2 A\lambda)\,q_{t-1} - \phi_1(1-A)\,\tilde{\pi}_{t-1} - \phi_2(1-A)\,\tilde{x}_{t-1}}{1+\bar{r}+\pi^T+\phi_1(1-A)\,\tilde{\pi}_{t-1}+\phi_2(1-A)\,\tilde{x}_{t-1}+(\phi_2 A\lambda+\phi_3(1-A))\,q_{t-1}}
\end{aligned}
\tag{C.1}
$$

Of course, the former dynamical system (with no LAW) can be obtained from the dynamical system (C.1) by setting $\phi_3 = 0$. Therefore, in the following, we focus on the dynamical system in (C.1). The results for the no-LAW case are obtained by setting $\phi_3 = 0$.

Consider the system (C.1), which is written in deviations from the targeted output gap and inflation. It is easy to verify that $\tilde{E}^T = (0,0,0)$ is a steady state of the system. This corresponds to the targeted equilibrium $E^T = \left(x^T, \pi^T, 0\right)$ for the original system.

By straightforward calculations, we obtain that the Jacobian matrix of system (C.1) in an arbitrary point $(\tilde{x}, \tilde{\pi}, q)$ is given by $J(\tilde{x}, \tilde{\pi}, q)$ which is

$$\begin{pmatrix} (1-A)\left(1-\frac{\phi_2}{\sigma}\right) & (1-A)\frac{1-\phi_1}{\sigma} & A\lambda\left(1-\frac{\phi_2}{\sigma}\right)-(1-A)\frac{\phi_3}{\sigma} \\ \kappa(1-A)\left(1-\frac{\phi_2}{\sigma}\right) & (1-A)\left(\beta+\kappa\frac{1-\phi_1}{\sigma}\right) & \kappa\left(A\lambda\left(1-\frac{\phi_2}{\sigma}\right)-(1-A)\frac{\phi_3}{\sigma}\right) \\ -D(1-A)\phi_2((1-A)q+B) & -D(1-A)\phi_1((1-A)q+B) & J_{3,3} \end{pmatrix}$$

where we introduced auxiliary constants

$$\begin{aligned} B &= 1+\bar{r}+\pi^T, \\ D &= \Big[(1-A)(\tilde{\pi}\phi_1-\lambda q\phi_2+q\phi_3+\tilde{x}\phi_2)+B+\lambda q\phi_2)\Big]^2, \\ J_{3,3} &= \frac{1}{D}\Big[(1-A)^2(\tilde{\pi}\phi_1+\tilde{x}\phi_2)+(1-A)B(1-\phi_3)-AB\lambda\phi_2\Big]. \end{aligned} \tag{C.2}$$

In particular, at the targeted equilibrium $\tilde{E}^T$, the constant $D = B^2$, and so the Jacobian matrix above becomes $J(0,0,0) \equiv J_0$, which is

$$J_0 = \begin{pmatrix} (1-A)\left(1-\frac{\phi_2}{\sigma}\right) & (1-A)\frac{1-\phi_1}{\sigma} & A\lambda\left(1-\frac{\phi_2}{\sigma}\right)-(1-A)\frac{\phi_3}{\sigma} \\ \kappa(1-A)\left(1-\frac{\phi_2}{\sigma}\right) & (1-A)\left(\beta+\kappa\frac{1-\phi_1}{\sigma}\right) & \kappa\left(A\lambda\left(1-\frac{\phi_2}{\sigma}\right)-(1-A)\frac{\phi_3}{\sigma}\right) \\ -\frac{(1-A)\phi_2}{B} & -\frac{(1-A)\phi_1}{B} & \frac{(1-A)(1-\phi_3)-A\lambda\phi_2}{B} \end{pmatrix}$$

It follows that the characteristic equation associated with the Jacobian matrix $J(0,0,0)$ is given by

$$\mu^3 + a_1\mu^2 + a_2\mu + a_3 = 0 \tag{C.3}$$

where coefficients a_1, a_2 and a_3 are calculated as

$$
\begin{aligned}
a_1 &= (1-A)\left(\frac{\kappa(\phi_1-1)+\phi_2}{\sigma} - \beta - 1 + \frac{\phi_3 - 1 - \lambda\phi_2}{B}\right) + \frac{\lambda\phi_2}{B} \\
a_2 &= -\frac{(1-A)^2\Big((\beta+1)(\phi_3-1) - \beta\lambda\phi_2 - \beta B + \kappa\lambda\phi_1\Big)}{B} \\
&\quad -\frac{(1-A)^2\Big((\beta B+1)\phi_2 + \kappa(-\lambda\phi_2+\phi_1+\phi_3-1)\Big)}{B\sigma} \\
&\quad -\frac{(1-A)\,\lambda\Big(\beta\sigma\phi_2 + \kappa(\phi_2 - \sigma\phi_1)\Big)}{B\sigma} \\
a_3 &= (1-A)^3\,\frac{\beta}{B}\left(\phi_3 - 1 + \frac{\phi_2}{\sigma}\right)
\end{aligned}
\tag{C.4}
$$

Then, the stability of the targeted equilibrium can be derived by imposing the conditions derived in Gardini et al. (2021) for the local stability of equilibrium of a three-dimensional dynamical system in discrete time. These conditions are:

$$
\begin{aligned}
1 + a_1 + a_2 + a_3 &> 0 \\
1 - a_1 + a_2 - a_3 &> 0 \\
1 - a_2 - a_3^2 + a_1 a_3 &> 0 \\
|a_3| &< 1
\end{aligned}
\tag{C.5}
$$

Moreover, the violation of the first condition leads to the bifurcation with eigenvalue $+1$, the violation of the second condition leads to the bifurcation with eigenvalue -1, the violation of the third condition leads to the bifurcation where a pair of complex-conjugate eigenvalues cross the unit circle.

Note that in the special case $\phi_1 = \phi_2 = \phi_3 = 0$, the dynamical system is linear in the state variables so that equilibrium $(0,0,0)$ is unique. The Jacobian matrix in this equilibrium becomes

$$J(0,0,0)=\begin{pmatrix} 1-A & \dfrac{1-A}{\sigma} & \lambda A \\ \kappa\,(1-A) & (1-A)\left(\beta+\dfrac{\kappa}{\sigma}\right) & A\kappa\lambda \\ 0 & 0 & \dfrac{1-A}{B} \end{pmatrix}. \tag{C.6}$$

Therefore, one of its eigenvalues is

$$\mu_1=\frac{1-A}{B}.$$

Note that B defined in (C.2) is positive and therefore $\mu_1\in[0,1)$. The other two eigenvalues are the eigenvalues of the matrix

$$\begin{pmatrix} 1-A & \dfrac{1-A}{\sigma} \\ \kappa\,(1-A) & (1-A)\left(\beta+\dfrac{\kappa}{\sigma}\right) \end{pmatrix}=(1-A)\begin{pmatrix} 1 & \dfrac{1}{\sigma} \\ \kappa & \beta+\dfrac{\kappa}{\sigma} \end{pmatrix} \tag{C.7}$$

We use this to prove Proposition 3.1.

ZLB is binding. Following Hommes and Lustenhouwer (2019b), when the ZLB is binding there exists a unique equilibrium, given by

$$\begin{aligned} x_{ZLB}&=\frac{\sigma((1-A)\,\beta-1)(-Ax^T-\lambda(\pi^T+\bar{r}))+\pi^T+\bar{r}-(1-A)\,\pi^T-(1-A)\,\beta\bar{r}}{(1-A)^2\,\beta\sigma-(1-A)\,(\beta\sigma+\kappa+\sigma)+\sigma} \\ \pi_{ZLB}&=\frac{A^2\beta\pi^T\sigma+\kappa(-(1-A)\,\pi^T-(1-A)\,\sigma x^T+\lambda\pi^T\sigma+\pi^T+\sigma x^T)+\bar{r}(\kappa\lambda\sigma+\kappa)}{(1-A)^2\,\beta\sigma-(1-A)\,(\beta\sigma+\kappa+\sigma)+\sigma} \\ q_{ZLB}&=\frac{\bar{r}+\pi^T}{A} \end{aligned} \tag{C.8}$$

Notice that by construction, the Jacobian matrix at the ZLB equilibrium is the same as at the targeted equilibrium E^T when $\phi_1=\phi_2=\phi_3=0$. Therefore, when the ZLB is applied, the stability region of the ZLB equilibrium coincides with the stability region of equilibrium E^T when no active monetary policy is in place, see Proposition 3.1. When condition (16) is violated, the ZLB equilibrium is clearly a saddle point. As we want to study the stabilizing effect of the monetary policy, we focus on configurations of the values of the parameters for which the targeted equilibrium is unstable in case of no active monetary policy. Therefore, also the ZLB equilibrium is unstable.

D Proofs of Propositions

Proof of Proposition 3.1. For $\phi_1 = \phi_2 = \phi_3 = 0$, the system (C.1) is linear. Then E^T is the unique equilibrium and it is either globally stable or the dynamics are unbounded. Moreover, the characteristic equation in (C.3) becomes

$$\left(\frac{1-A}{B} - \mu\right)\left(\mu^2 - \frac{(1-A)(\beta\sigma + \kappa + \sigma)}{\sigma}\mu + (1-A)^2\beta\right) = 0 \tag{D.1}$$

Therefore, the eigenvalues are

$$\mu_1 = \frac{1-A}{B} = \frac{1-A}{1+\bar{r}+\pi^T},$$

where we used B defined in (C.2), and

$$\mu_{\pm} = \frac{(1-A)\left(\beta\sigma + \kappa + \sigma \pm \sqrt{\kappa^2 + 2(\beta+1)\kappa\sigma + (\beta-1)^2\sigma^2}\right)}{2\sigma}$$

As $B > 0$ and $A \in (0, 1]$, the first eigenvalue $\mu_1 \in (0, 1]$. The two other eigenvalues are always real, because $\kappa^2 + 2(\beta+1)\kappa\sigma + (\beta-1)^2\sigma^2 > 0$. Moreover, $\mu_+ > |\mu_-| \geq 0$. Therefore, both eigenvalues are within the unit circle whenever $\mu_+ < 1$. Using that $A, \beta \in (0, 1]$, condition (16) follows. By standard calculations, we can prove that A^* in (16) is always smaller than 1. This completes the proof. □

Proof of Proposition 3.2. Note that $\alpha = \infty$ is equivalent to $\phi_3 = 0$. Moreover, for $\lambda = \phi_3 = 0$ the dynamics of q depends on x and π, but the dynamics of x and π do not depend on q. In this case, the equations providing the dynamics of x and π are called master equations and the equation providing the dynamics of q is called slave equation. The dynamical system made by the two master equations is as the one considered in Hommes and Lustenhouwer (2019b), for which the same stability conditions apply. Once there is convergence on x and on π to x^T and π^T, respectively, the slave equation is such that q converges to 0. This proves the first point of the Proposition. For $\lambda > 0$, the system is not of the slave-master type and, solving for $x_t = x_{t+1} = x^*$, $\pi_t = \pi_{t+1} = \pi^*$ and $q_t = q_{t+1} = q^*$, straightforward algebra shows that E^T and E^{NT} are equilibria of the model. In E^T the ZLB never applies, instead, in E^{NT} the ZLB can apply. However, when the ZLB applies, the dynamical system becomes a linear system with a unique equilibrium different from E^{NT}, see the details in Appendix C. Therefore, E^{NT} is not an equilibrium of the model when it is in the ZLB region. Numerical analysis reveals that these two equilibria cannot be stable at the same time. □

Proof of Proposition B.1. For $\alpha = 0$, the dynamical system (C.1) applies outside the ZLB region. Then, solving the system for $x_t = x_{t+1} = x^*$, $\pi_t = \pi_{t+1} = \pi^*$ and $q_t = q_{t+1} = q^*$, straightforward algebra show that E^T and E^{LAW} are equilibria of the model. Consider the stability conditions (system of four inequalities) for E^T derived in (C.4)-(C.5). They indicate that the eigenvalues of E^T can take value +1 (i.e, bifurcation of eigenvalue +1), value −1 (i.e, bifurcation of eigenvalue −1), or being complex-conjugate and with modulus equal to one (i.e, Neimark-Sacker bifurcation). Setting $\phi_3 = A\,(\sigma - \phi_2)$, the Jacobian matrix in E^T, derived in (C), becomes

$$J\left(E^T\right) := \begin{pmatrix} (1-A)\left(1-\dfrac{\phi_2}{\sigma}\right) & \dfrac{(1-A)\,(1-\phi_1)}{\sigma} & 0 \\ \dfrac{(1-A)\,\kappa(\sigma-\phi_2)}{\sigma} & 0 & \\ -\dfrac{(1-A)\,\phi_2}{B} & -\dfrac{(1-A)\,\phi_1}{B} & \dfrac{1-A-\lambda\sigma A}{B} \end{pmatrix} \tag{D.2}$$

Then, two eigenvalues $\mu_{1,2}$ solve

$$\det \begin{pmatrix} 1 - A - \dfrac{(1-A)\,\phi_2}{\sigma} - \mu & \dfrac{(1-A)\,(1-\phi_1)}{\sigma} \\ \dfrac{(1-A)\,\kappa(\sigma-\phi_2)}{\sigma} & \dfrac{(1-A)\,(\beta\sigma+\kappa\,(1-\phi_1))}{\sigma} - \mu \end{pmatrix} = 0 \qquad \text{(D.3)}$$

and are as in Hommes and Lustenhouwer (2019b), that is, they are in modulus less than one when the conditions (19) in Proposition 3.2 are satisfied, while the third eigenvalue is $\mu_3 = (1 - A - \lambda\sigma A)/B$ and it is in modulus less than one when $\lambda < 1/\sigma$. This completes the proof. □

Proof of Proposition 3.3. For $\alpha \in (0, +\infty)$ and $\phi_3 > 0$, we have that around the targeted equilibrium E^T, where $q = 0$, the dynamical system (which does not depend on ϕ_3) in (C) applies. This is the dynamical system that applies also when the LAW policy is never active. Therefore, the stability of the targeted equilibrium E^T is not affected by ϕ_3 and the same results of Proposition 3.2 apply regarding its stability. Since the dynamical system in (C) admits also the equilibrium E^{NT}. Existence and stability of this equilibrium for the dynamical system in (C) are discussed in Proposition 3.2 and they hold also here as long as E^{NT} is in the region where the dynamical system in (C) applies, that is if $|q^*| < \alpha$. In the region where $|q^*| > \alpha$, either the ZLB is binding or the dynamical system in (C.1). The equilibria of this dynamical system and their stability are discussed in Proposition B.1. Here, let us note that only E^{LAW} can be an equilibrium of the dynamical system in (C.1) that lies in the region $|q^*| > \alpha$. This completes the proof. □