Comparative statics with adjustment costs and the le Chatelier principle*

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Abstract: We develop a theory of monotone comparative statics in the presence of adjustment costs. We show that comparative-statics conclusions may be drawn under the usual ordinal complementarity assumptions on the objective function, assuming almost nothing about costs: the only requirement is that non-adjustment be costless. We use this insight to provide a general treatment of the le Chatelier principle based on adjustment costs. We extend these results to a fully dynamic, forward-looking model of adjustment over time: given only minimal structure on costs, optimal adjustment follows a monotone path. We apply our results to models of investment and of sticky prices.

Keywords: Adjustment costs, comparative statics, le Chatelier

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1 Introduction

Adjustment costs play a major role in explaining a wide range of economic phenomena. Examples include the investment behavior of firms (e.g. Jorgenson, 1963; Hayashi, 1982; Cooper & Haltiwanger, 2006), price stickiness (e.g. Mankiw, 1985; Caplin & Spulber, 1987; Golosov & Lucas, 2007), aggregate consumption dynamics (e.g. Kaplan & Violante, 2014; Berger & Vavra, 2015) and housing consumption and asset pricing (Grossman & Laroque, 1990).

In this paper, we develop a theory of monotone comparative statics with adjustment costs. Our fundamental insight is that, surprisingly, very little needs to be assumed about the cost function: comparative statics requires only that *not* adjusting be costless, plus the usual ordinal complementarity assumptions on the objective function. We use this insight to show that Samuelson's (1947) *le Chatelier principle* is far more general than previously thought: it holds whenever adjustment is costly, given only minimal structure on costs. We extend our comparative-statics and le Chatelier results to a fully forward-looking dynamic model of adjustment.

We apply our results to models of factor demand, capital investment, and pricing. These models are typically studied only under strong functional-form assumptions, and the cases of convex and non-convex costs are considered separately and handled very differently. Our general results yield robust comparative statics for these standard models, dispensing with auxiliary assumptions and handling convex and non-convex costs in a unified fashion.

The abstract setting is as follows. An agent chooses an action x from a sublattice $L \subseteq \mathbb{R}^n$. Her objective $F(x, \theta)$ depends on a parameter θ. At the initial parameter $\underline{\theta}$, the agent chose $\underline{x} \in \arg\max_{x \in L} F(x, \underline{\theta})$. The parameter now increases to $\bar{\theta} > \underline{\theta}$, and the agent may adjust her choice. Adjusting dimension i of the action by $\epsilon_i = x_i - \underline{x}_i$ costs $C_i(\epsilon_i) \geqslant 0$, and the agent's new choice maximizes $G(x, \bar{\theta}) = F(x, \bar{\theta}) - \sum_{i=1}^n C_i(x_i - \underline{x}_i)$.

The cost functions $C_i : \mathbb{R} \to [0, \infty]$ are quite general: our only assumptions are that non-adjustment is costless ($C_i(0) = 0$) and, for some results, that C_i is single-dipped. Thus costs need not be convex, for example. Some adjustments ϵ_i may be infeasible, as captured by a prohibitive cost $C_i(\epsilon_i) = \infty$.

Our basic question is under what assumptions on the objective F and costs C_i the agent's choice increases, in the sense that $\bar{x} \geqslant \underline{x}$ for some $\bar{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$ (provided the argmax is not empty). Our fundamental result, Theorem 1, answers this question: *nothing* need be assumed about costs except that non-adjustment is free ($C_i(0) = 0$), while F need only satisfy the ordinal complementarity conditions of *quasi-supermodularity* and *single-crossing differences* that feature in similar comparative-statics results absent adjustment costs (see Milgrom & Shannon, 1994). Thus costs need not even be single-dipped, and the objective need not satisfy any cardinal properties, such as supermodularity or increasing differences. We also give a 'strict' variant (Proposition 1): adding either of two mild assumptions yields the stronger conclusion that $\bar{x} \geqslant \underline{x}$ for *every* $\bar{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$.

We use our fundamental result to re-think Samuelson's (1947) *le Chatelier principle,* which asserts that the response to a parameter shift is greater at longer horizons. Our Theorem 2 provides that the le Chatelier principle holds whenever short-run adjustment is subject to single-dipped adjustment costs C_i, long-run adjustment is frictionless, and the objective F satisfies the ordinal complementarity conditions. Formally, the theorem states that under these assumptions, given any long-run choice $\bar{x} \in \arg\max_{x \in L} F(x, \bar{\theta})$ satisfying $\bar{x} \geqslant \underline{x}$, we have $\bar{x} \geqslant \hat{x} \geqslant \underline{x}$ for some optimal short-run choice $\hat{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$, provided the argmax is nonempty.[1] This substantially generalizes Milgrom and Roberts's (1996) le Chatelier principle, in which short-run adjustment is assumed to be impossible for some dimensions i ($C_i(\epsilon_i) = \infty$ for all $\epsilon_i \neq 0$) and costless for the rest ($C_i \equiv 0$). We extend our le Chatelier principle to the case in which long-run adjustment is also costly, but cheaper on the margin than short-run adjustment (Proposition 2).

We then extend our comparative-statics and le Chatelier theorems to a fully dynamic, forward-looking model of costly adjustment over time. The parameter permanently shifts at date $t = 0$ from $\underline{\theta}$ to $\bar{\theta} > \underline{\theta}$, shifting the frictionless optimum from $\underline{x}$ to some $\bar{x} \geqslant \underline{x}$. Starting at $x_0 = \underline{x}$, the agent chooses an adjustment path $(x_t)_{t=1}^{\infty}$ to maximize a discounted sum of her period payoffs $F(x_t, \bar{\theta}) - \sum_{i=1}^{n} C_i(x_{t,i} - x_{t-1,i})$. Theorem 3 validates the le Chatelier principle: the agent adjusts more at longer horizons, in the sense that $\underline{x} \leqslant x_t \leqslant x_T \leqslant \bar{x}$ holds at any dates $t < T$, for some optimal path $(x_t)_{t=1}^{\infty}$ (provided an optimal path exists). The hypotheses of this theorem strengthen those of Theorem 2 by requiring *cardinal* complementarity of F (supermodularity and increasing differences), as well as a mild technical condition. With time-varying costs C_i^t, the same hypotheses imply the weaker conclusion that $\underline{x} \leqslant x_t \leqslant \bar{x}$ for every period t (Proposition 3).

The rest of this paper is arranged as follows. In the next section, we de-

[1]Furthermore, if $\bar{x}$ is the largest element of $\arg\max_{x \in L} F(x, \underline{x})$, then $\bar{x} \geqslant \hat{x}$ for *any* short-run choice $\hat{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$ such that $\hat{x} \geqslant \underline{x}$.

scribe the setting. We present our fundamental comparative-statics insight (Theorem 1) in section 3. In section 4, we develop a general le Chatelier principle (Theorem 2), and apply it to monopoly pricing and factor demand. In section 5, we introduce a dynamic, forward-looking adjustment model, derive a dynamic le Chatelier principle (Theorem 3), and apply it to pricing. We conclude in section 6 by examining the implications of our results for capital investment. The appendix contains a number of definitions of standard order-theoretic concepts, as well as all proofs omitted from the main text.

2 Setting

The agent's long-term objective is $F(x, \theta)$, where x is the choice variable and $\theta \in \Theta$ a parameter. The choice variable x belongs to a sublattice L of $\mathbb{R}^n$; more generally, it could be a sublattice of any vector lattice.

We shall be assuming that the dimensions x_i of the action x are complementary: precisely, that the function $F(\cdot, \theta)$ is quasi-supermodular (or, sometimes, supermodular). This is automatically the case when the action is one-dimensional ($n = 1$), as is frequently the case in applications.

At the initial parameter $\theta = \underline{\theta}$, an optimal choice $\underline{x}$ was made:

$$\underline{x} \in \arg\max_{x \in L} F(x, \underline{\theta}).$$

(Note that we allow for a multiplicity of optimal actions.) This is the agent's 'starting point,' and we shall consider how she responds in the short and long run to a change in the parameter from $\underline{\theta}$ to $\bar{\theta}$, where $\underline{\theta} < \bar{\theta}$.

Each dimension x_i of the choice variable is subject to an adjustment cost $C_i : \mathbb{R} \to [0, \infty]$, so that adjusting from $\underline{x}_i$ to x_i costs $C_i(x_i - \underline{x}_i)$. Our only maintained assumption on C_i is that non-adjustment is free: $C_i(0) = 0$. Note that we allow some adjustments ϵ_i to have infinite cost $C_i(\epsilon_i) = \infty$, meaning that they are infeasible.

Given a vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ of adjustments, the total cost of adjustment is

$$C(\epsilon) = \sum_{i=1}^{n} C_i(\epsilon_i).$$

The agent adjusts her choice $x \in L$ to maximize

$$G(x, \bar{\theta}) = F(x, \bar{\theta}) - C(x - \underline{x}).$$

Example 1. *(a)* $C_i(\epsilon_i) = |\epsilon_i|$. *(b) Free disposal:* $C_i(\epsilon_i) = \epsilon_i$ *if* $\epsilon_i > 0$ *and* $C_i(\epsilon_i) = 0$ *if* $\epsilon_i \leqslant 0$. *(c) A constraint* $\mathcal{E} \subseteq \mathbb{R}$*:* $C_i(\epsilon_i) = 0$ *if* $\epsilon_i \in \mathcal{E}$ *and* $C_i(\epsilon_i) = \infty$ *otherwise. (d) Increments:* $C_i(\epsilon_i) = |\epsilon_i|$ *if* ϵ_i *is integer and* $C_i(\epsilon_i) = \infty$ *otherwise. (e) Minimum adjustments* $\underline{\epsilon}_i \leqslant 0 \leqslant \bar{\epsilon}_i$*:* $C_i(0) = 0$, $C_i(\epsilon_i) = \infty$ *if* $\epsilon_i \neq 0$ *and* $\underline{\epsilon}_i < \epsilon_i < \bar{\epsilon}_i$, *and* $C_i(\epsilon_i) = |\epsilon_i|$ *otherwise.*

3 Comparative statics

Our fundamental comparative-statics result is the following.

Theorem 1. *Suppose* $F(x, \theta)$ *is quasi-supermodular in* x *and has single-crossing differences in* (x, θ)*, and that* $C_i(0) = 0$ *for every dimension* i*. Then* $\hat{x} \geqslant \underline{x}$ *for some* $\hat{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$*, provided the argmax is nonempty.*

Proof. Let $x' \in \arg\max_{x \in L} G(x, \bar{\theta})$. We claim that $\hat{x} = \underline{x} \vee x'$ also maximizes $G(\cdot, \bar{\theta})$; obviously this is bigger than $\underline{x}$. We have $F(\underline{x}, \underline{\theta}) \geqslant F(\underline{x} \wedge x', \underline{\theta})$ by definition of $\underline{x}$. Thus $F(\underline{x} \vee x', \underline{\theta}) \geqslant F(x', \underline{\theta})$ by quasi-supermodularity, whence $F(\underline{x} \vee x', \bar{\theta}) \geqslant F(x', \bar{\theta})$ by single-crossing differences. Since each C_i satisfies $C_i(0) = 0 \leqslant C_i(\epsilon_i)$ for any adjustment ϵ_i, we have

$$C_i\left(\max\{\underline{x}_i, x'_i\} - \underline{x}_i\right) \leqslant C_i\left(x'_i - \underline{x}_i\right) \quad \text{for each dimension } i,$$

and thus $C(\underline{x} \vee x' - \underline{x}) \leqslant C(x' - \underline{x})$. (That was the crucial step.) Thus

$$G(\hat{x}, \bar{\theta}) = F(\underline{x} \vee x', \bar{\theta}) - C(\underline{x} \vee x' - \underline{x}) \geqslant F(x', \bar{\theta}) - C(x' - \underline{x}) = G(x', \bar{\theta}).$$

Since x' maximizes $G(\cdot, \bar{\theta})$ on L, it follows that $\hat{x}$ does, too. **QED**

Theorem 1 shows that the basic comparative-statics result (see Milgrom & Shannon, 1994, Theorem 4), which guarantees comparative statics given quasi-supermodularity and single-crossing differences of the objective F, is not upset by the presence of adjustment costs: an increase of the parameter θ still leads to a higher action (modulo tie-breaking).

This result is striking because the assumptions on the adjustment cost C are weak. We assumed only additive separability and that non-adjustment is free. Even these assumptions are stronger than necessary; the exact property that C must satisfy is that for any vector ϵ, we have

$$C(\epsilon_1, \ldots, \epsilon_n) \geqslant C\left(\max\{\epsilon_1, 0\}, \ldots \max\{\epsilon_n, 0\}\right).$$

(In the proof, $\epsilon = x' - \underline{x}$, and $(\max\{\epsilon_1, 0\}, \ldots, \max\{\epsilon_n, 0\}) = \underline{x} \vee x' - \underline{x}$.)

It is also worth noting that the objective F need not satisfy any cardinal properties, such as supermodularity or increasing differences: the ordinal properties of quasi-supermodularity and single-crossing differences suffice. This is valuable for applications, since it is common for F to exhibit single-crossing but not increasing differences. That is the case in monopoly pricing, for example—see section 4.2 below.

Remark 1. *Theorem 1 is phrased differently than the basic result (see Milgrom & Shannon, 1994, Theorem 4), which asserts that if $F(x, \theta)$ is quasi-supermodular in x and has single-crossing differences in (x, θ), then*

$$\begin{aligned} & x'' \in \arg\max_{x \in L} F(x, \underline{\theta}) \quad \textit{and} \quad x' \in \arg\max_{x \in L} F(x, \bar{\theta}) \\ \Longrightarrow \quad & x'' \wedge x' \in \arg\max_{x \in L} F(x, \underline{\theta}) \quad \textit{and} \quad x'' \vee x' \in \arg\max_{x \in L} F(x, \bar{\theta}). \end{aligned}$$

A version of Theorem 1 with this form also holds: under the same hypotheses,

$$\begin{aligned} & \underline{x} \in \arg\max_{x \in L} F(x, \underline{\theta}) \quad \textit{and} \quad x' \in \arg\max_{x \in L} G(x, \bar{\theta}) \\ \Longrightarrow \quad & \underline{x} \wedge x' \in \arg\max_{x \in L} F(x, \underline{\theta}) \quad \textit{and} \quad \underline{x} \vee x' \in \arg\max_{x \in L} G(x, \bar{\theta}). \end{aligned}$$

The latter property (concerning $\underline{x} \vee x'$) is exactly what is shown in the proof of Theorem 1. For the former property (concerning $\underline{x} \wedge x'$), suppose it were to fail; then $F(\underline{x}, \underline{\theta}) > F(\underline{x} \wedge x', \underline{\theta})$, so that replicating the steps in the proof of Theorem 1 delivers $G(\underline{x} \vee x', \bar{\theta}) > G(x', \bar{\theta})$, a contradiction with the optimality of x' for $G(\cdot, \bar{\theta})$.

3.1 'Strict' comparative statics

We now provide a 'strict' counterpart to Theorem 1, giving conditions under which $\hat{x} \geqslant \underline{x}$ holds for *any* optimal choice $\hat{x}$. Say that *reduction is costly* if $C_i(\epsilon_i) > 0$ for every $\epsilon_i < 0$ and every dimension i.

PROPOSITION 1. *Suppose $F(x, \theta)$ is quasi-supermodular in x and has single-crossing differences in (x, θ), that $C_i(0) = 0$ for every dimension i, and either*

(a) that the single-crossing differences of $F(x, \theta)$ in (x, θ) is strict, or

(b) that reduction is costly.

Then $\hat{x} \geqslant \underline{x}$ for any $\hat{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$.[2]

Proposition 1 is the costly-adjustment analog of the standard 'strict' comparative-statics result (see Milgrom & Shannon, 1994, Theorem 4′), which states that given any $\underline{x} \in \arg\max_{x \in L} F(x, \underline{\theta})$, if $F(x, \theta)$ is quasi-supermodular in x and has *strictly* single-crossing differences in (x, θ), then $\hat{x} \geqslant \underline{x}$ for any $\hat{x} \in \arg\max_{x \in L} F(x, \bar{\theta})$. Part (a) directly extends this result to the costly-adjustment case. Part (b) shows that the 'strictness' in the hypotheses required to obtain a 'strict' comparative-statics conclusion can come from the costs C_i rather than the objective F. In particular, the 'strict' hypothesis that reduction is costly ensures that even if some actions strictly lower than $\underline{x}$ are optimal for the gross objective $F(\cdot, \bar{\theta})$, they will not be chosen due to their cost.

[2] A variant of Proposition 1 can be obtained by mixing the 'strictness' properties (a) and (b): if $x = (y, z)$, where $F(y, z, \theta)$ has strict single-crossing differences in (y, θ) for any fixed z, and z-reduction is costly, then the conclusion goes through, with the same proof.

4 The le Chatelier principle

The le Chatelier principle asserts that long-run elasticities exceed short-run elasticities. In this section, we show that the le Chatelier principle is far more general than previously claimed: it arises whenever adjustment is costly, given only minimal structure on costs. The classic formalization, which assumes that only some dimensions of the action are adjustable in the short run, is the special case in which each dimension's adjustment cost is either prohibitively high or equal to zero.

We consider the agent's short- and long-run responses to an increase of the parameter from $\underline{\theta}$ to $\bar{\theta} > \underline{\theta}$. Her short-run response $\hat{x}$ takes adjustment costs into account, so it maximizes $G(\cdot, \bar{\theta})$. In the long run, the agent adjusts to a new frictionless optimum $\bar{x} \in \arg\max_{x \in L} F(x, \bar{\theta})$.

Say that the adjustment cost C_i is *upward monotone* if it is increasing on $\mathbb{R}_+$, i.e., $C_i(\epsilon_i) \leqslant C_i(\epsilon_i')$ whenever $0 \leqslant \epsilon_i \leqslant \epsilon_i'$. A sufficient condition is single-dippedness (which demands, in addition to upward monotonicity, that C_i be decreasing on $\mathbb{R}_-$).

Theorem 2 (le Chatelier principle). *Suppose $F(x, \theta)$ is quasi-supermodular in x and has single-crossing differences in (x, θ), and that each C_i is upward monotone with $C_i(0) = 0$. Let $\bar{x} \in \arg\max_{x \in L} F(x, \bar{\theta})$ satisfy $\bar{x} \geqslant \underline{x}$.*[3] *Then*

- *$\bar{x} \geqslant \hat{x} \geqslant \underline{x}$ for some $\hat{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$ provided the argmax is nonempty, and*

[3] Given the properties of F, a standard result (Milgrom & Shannon, 1994, Theorem 4) guarantees that such an $\bar{x}$ exists, provided the argmax is nonempty.

- *if* $\bar{x}$ *is the largest element of* $\arg\max_{x \in L} F(x, \bar{\theta})$*, then* $\bar{x} \geqslant \hat{x}$ *for any* $\hat{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$ *such that* $\hat{x} \geqslant \underline{x}$*.*

Proof. For the first part, assume that $\arg\max_{x \in L} G(x, \bar{\theta})$ is nonempty. By Theorem 1, we may choose an $x' \in \arg\max_{x \in L} G(x, \bar{\theta})$ such that $x' \geqslant \underline{x}$. We claim that $\bar{x} \wedge x'$ also maximizes $G(\cdot, \bar{\theta})$; this suffices since $\bar{x} \geqslant \bar{x} \wedge x' \geqslant \underline{x}$. We have $F(\bar{x} \vee x', \bar{\theta}) \leqslant F(\bar{x}, \bar{\theta})$ by definition of $\bar{x}$, which by quasi-supermodularity implies that $F(x', \bar{\theta}) \leqslant F(\bar{x} \wedge x', \bar{\theta})$. Since each C_i is upward monotone and $x' \geqslant \bar{x} \wedge x' \geqslant \underline{x}$, we have $C(x' - \underline{x}) \geqslant C(\bar{x} \wedge x' - \underline{x})$. Thus

$$F(x', \bar{\theta}) - C(x' - \underline{x}) \leqslant F(\bar{x} \wedge x', \bar{\theta}) - C(\bar{x} \wedge x' - \underline{x}),$$

which since x' maximizes $G(\cdot, \bar{\theta})$ on L implies that $\bar{x} \wedge x'$ does, too.

For the second part, let $\bar{x}$ be the largest element of $\arg\max_{x \in L} F(x, \bar{\theta})$, and let $\hat{x} \geqslant \underline{x}$ belong to $\arg\max_{x \in L} G(x, \bar{\theta})$; we will show that $\bar{x} \geqslant \hat{x}$. The optimality of $\hat{x}$ implies that $G(\hat{x}, \bar{\theta}) \geqslant G(\bar{x} \wedge \hat{x}, \bar{\theta})$. Since $\bar{x} \geqslant \underline{x}$ and $\hat{x} \geqslant \underline{x}$, the upward monotonicity of each C_i implies that $C(\hat{x} - \underline{x}) \geqslant C(\bar{x} \wedge \hat{x} - \underline{x})$. Thus $F(\hat{x}, \bar{\theta}) \geqslant F(\bar{x} \wedge \hat{x}, \bar{\theta})$, so that $F(\bar{x} \vee \hat{x}, \bar{\theta}) \geqslant F(\bar{x}, \bar{\theta})$ by quasi-supermodularity; hence $\bar{x} \vee \hat{x}$ also maximizes $F(\cdot, \bar{\theta})$ on L. Since $\bar{x}$ is the largest maximizer of $F(\cdot, \bar{\theta})$, we conclude that $\bar{x} \geqslant \hat{x}$. **QED**

Theorem 2 nests the le Chatelier principle of Milgrom and Roberts (1996), in which it is assumed that only some dimensions x_i of the choice variable can be adjusted in the short run, and that such adjustments are costless. This is the special case of our model in which some dimensions i have $C_i \equiv 0$, while the rest have $C_i(\epsilon_i) = \infty$ for every $\epsilon_i \neq 0$.

Like Theorem 1, Theorem 2 requires F only to satisfy ordinal complementarity properties, not cardinal ones. This greatly extends its applicability, allowing it to be used to study pricing, for example (see section 4.2 below).

4.1 Application to factor demand

Consider a stylized model of production, following Milgrom and Roberts (1996). A firm uses capital k and labor ℓ to produce output $f(k, \ell)$. Profit at real factor prices (r, w) is $F(k, \ell, -w) = f(k, \ell) - rk - w\ell$. Adjustment costs C_i are single-dipped with $C_i(0) = 0$, and otherwise unrestricted.

Theorem 2 provides that if capital and labor are complements (i.e., f is supermodular), then a drop in the wage w precipitates a short-run increase of ℓ (and of k), and a further increase in the long run. If capital and labor are substitutes (f submodular), then we may apply Theorem 2 to the choice variable $(x_1, x_2) = (-k, \ell)$ to conclude that ℓ still increases (but k *decreases*) in the short run, and then increases further in the long run.[4]

Milgrom and Roberts (1996) were the first to use the theory of monotone comparative statics to obtain such a result. They assumed that labor adjustments are costless ($C_\ell \equiv 0$) and that capital cannot be adjusted at all in the short run ($C_k(\epsilon_k) = \infty$ for every $\epsilon_k \neq 0$). Our result reveals that much weaker assumptions suffice. It turns out not to matter whether labor is cheap to adjust relative to capital. What matters is, rather, that short-run adjustments are costly and that long-run adjustments are not.[5]

[4]This trick is due to Milgrom and Roberts (1996). Applying Theorem 2 here requires that the capital adjustment cost be decreasing on $\mathbb{R}_-$; single-dippedness ensures this.

[5]More generally, long-run adjustment can be costly, so long as it is cheaper on the

4.2 Application to pricing

The central plank of new Keynesian macroeconomic models is price stickiness, and the oldest and most important microfoundation for this property is (non-convex) adjustment costs (e.g. Mankiw, 1985; Caplin & Spulber, 1987; Golosov & Lucas, 2007). These may be real costs of updating what prices are displayed: empirically, such 'menu costs' can be non-negligible (see e.g. Levy, Bergen, Dutta, & Venable, 1997). Or they may arise from consumers reacting adversely to price hikes by temporarily reducing demand (as in Antić & Salant, in progress).

To study pricing, we consider the simplest model, following Milgrom and Roberts (1990): a monopolist with constant marginal cost $c \geqslant 0$ faces a decreasing demand curve $D(\cdot, \eta)$ parametrized by η, so earns a profit of $F(p, (c, -\eta)) = (p - c)D(p, \eta)$ if she prices at $p \in \mathbb{R}_+$. We assume that demand $D(p, \eta)$ is always strictly positive, and that η is an elasticity shifter: when it increases, so does the absolute elasticity of demand at every price p. Then profit $F(p, \theta) = F(p, (c, -\eta))$ has log increasing differences in $(p, -\eta)$ and has increasing differences in (p, c), so has single-crossing differences in $(p, \theta) = (p, (c, -\eta))$. $F(p, \theta)$ is automatically quasi-supermodular in p since it is one-dimensional ($n = 1$).

Adjusting the price by ϵ incurs a cost of $C(\epsilon) \geqslant 0$. We assume nothing about C except that $C(0) = 0$. In many macroeconomic models, it is a pure fixed cost: $C(\epsilon) = k > 0$ for every $\epsilon \neq 0$. When adjustment costs arise from price-hike-averse consumers, we have $C(\epsilon) = 0$ for $\epsilon \leqslant 0$ and $C(\epsilon) > 0$ for

margin than short-run adjustment—see section 4.3 below.

$\epsilon > 0$. If consumers are inattentive to small price changes, then $C(\epsilon) = 0$ if $\epsilon \in [\underline{\epsilon}, \bar{\epsilon}]$ and $C(\epsilon) > 0$ otherwise, where $\underline{\epsilon} < 0 < \bar{\epsilon}$.

By Theorem 1, the familiar comparative-statics properties of the monopoly problem are highly robust to the introduction of adjustment costs: it remains true that the monopolist raises her price whenever her marginal cost c rises and whenever demand becomes less elastic (i.e., η falls). No assumptions on the adjustment cost C are required for these results except that $C(0) = 0$.

Under the mild additional assumption that C is upward monotone, Theorem 2 yields a dynamic prediction: in response to a shock that increases her marginal cost or decreases the elasticity of demand, the monopolist initially raises her price, and then increases it further over the longer run. Thus one-off permanent cost and demand shocks lead, quite generally, to price increases that are to some extent gradual.

A key reason why we can draw such general conclusions about pricing is that Theorems 1 and 2 require F to satisfy only ordinal (not cardinal) complementarity conditions. Specifically, we used the fact that the monopolist's profit undergoes a 'single-crossing differences' shift when demand becomes less elastic (i.e., when η falls). A result which assumed the cardinal property of *increasing* differences would have been inapplicable, since elasticity shifts do not generally cause profit to shift in an 'increasing differences' fashion.[6]

[6]This applied advantage of requiring only ordinal complementarity was emphasized by Milgrom and Roberts (1990) and Milgrom and Shannon (1994) in the context of models with costless adjustment.

4.3 Short- vs. medium-run adjustment

We now extend our le Chatelier principle to assert not merely that long-run elasticities exceed short-run elasticities, but further that medium-run elasticities lie in-between.

To that end, we allow the cost $C_i(\cdot, h) : \mathbb{R} \to [0, \infty]$ of adjusting dimension i to vary with the horizon h. We shall assume that C_i has decreasing differences, meaning that adjustment is cheaper on the margin at longer horizons. At horizon h, the agent adjusts her choice $x \in L$ to maximize $G(x, \bar{\theta}, h) = F(x, \bar{\theta}) - C(x - \underline{x}, h)$, where $C(\epsilon, h) = \sum_{i=1}^n C_i(\epsilon_i, h)$. We compare a short horizon $\underline{h}$ with a longer horizon $\bar{h} > \underline{h}$.

By applying Theorem 1, Topkis's theorem and Theorem 2 in turn, we obtain a chain of three inequalities:

Proposition 2. *Suppose $F(x, \theta)$ is supermodular in x and has increasing differences in (x, θ), and that each $C_i(\epsilon_i, h)$ has decreasing differences in (ϵ_i, h), is upward monotone in ϵ_i, and satisfies $C_i(0, h) = 0$. Then*

$$\begin{aligned} \underline{x} &\leqslant \max\left\{\arg\max_{x \in L} G(x, \bar{\theta}, \underline{h})\right\} \\ &\leqslant \max\left\{\arg\max_{x \in L} G(x, \bar{\theta}, \bar{h})\right\} \leqslant \max\left\{\arg\max_{x \in L} F(x, \bar{\theta})\right\} \end{aligned}$$

for any horizons $\underline{h} < \bar{h}$, provided all three maxima exist.[7]

[7] Sufficient conditions for these maxima to exist are compactness of L, upper semi-continuity of $F(\cdot, \bar{\theta})$ and lower semi-continuity of $C(\cdot, \underline{h})$ and $C(\cdot, \bar{h})$. Then all three argmaxes are nonempty and compact. They are also sublattices, by a standard result (Topkis, 1978, Theorem 4.1) which applies since $F(\cdot, \bar{\theta})$ is supermodular and $C(\cdot, \underline{h})$ and $C(\cdot, \bar{h})$ are additively separable (so that $G(\cdot, \bar{\theta}, \underline{h})$ and $G(\cdot, \bar{\theta}, \bar{h})$ are supermodular, too). It follows by the Frink–Birkhoff theorem (see Topkis, 1998, Theorem 2.3.1) that all three argmaxes are subcomplete sublattices, so possess greatest elements.

Proposition 2 admits alternative interpretations. One is that $\bar{h}$ is the *long* run, in which case the result says that the le Chatelier principle remains valid if long-run adjustment is also costly, provided it is cheaper on the margin than adjusting in the short run. Another interpretation is that h is a parameter governing adjustment costs; the result then says that short-run adjustments are greater the lower are marginal costs.

Note that unlike our preceding results, Proposition 2 requires cardinal assumptions (supermodularity and monotone differences) rather than merely ordinal ones (quasi-supermodularity and single-crossing differences). This is because the middle inequality demands comparative statics with respect to the parameter h, which essentially requires that $G(x, \bar{\theta}, h) = F(x, \bar{\theta}) - C(x - \underline{x}, h)$ be quasi-supermodular in x and have single-crossing differences in (x, h) (see Milgrom & Shannon, 1994, Theorem 4). For this, it is not enough that F and C_i themselves satisfy the ordinal complementarity conditions, since these properties are not generally preserved by addition. By contrast, the stronger cardinal complementarity conditions *are* stable under addition.

This logic is a commonplace of the literature: when the objective is the sum of two or more primitive functions, comparative-statics conclusions can typically be drawn only under *cardinal* complementarity assumptions, since ordinal properties are not preserved by summation. What is noteworthy is thus not that Proposition 2 requires cardinal assumptions, but that our preceding results (Theorems 1 and 2, and Proposition 1) do not.

5 Dynamic adjustment

The le Chatelier principle takes a classical, 'reduced-form' approach to dynamics. In this section, we instead consider a fully-fledged, forward-looking dynamic model of adjustment. We show that the le Chatelier principle remains valid: in the short run, the agent's choice exceeds $\underline{x}$ and does not overshoot $\bar{x}$. We furthermore show that the path of adjustment is monotone, so that the agent adjusts more in the medium run than in the short run.

5.1 Setting

The agent faces an infinite-horizon decision problem in discrete time. In each period, she takes an action $x \in L$, and earns a payoff of $F(x, \theta)$ that is supermodular in x and has increasing differences in (x, θ). The cost of adjusting dimension i by ϵ_i from period $t-1$ to period t is $C_i^t(\epsilon_i) \in [0, \infty]$. The total cost of an adjustment $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ is denoted $C^t(\epsilon) = \sum_{i=1}^n C_i^t(\epsilon_i)$.

At the outset, there is a one-time permanent shift of the parameter from $\underline{\theta}$ to $\bar{\theta} > \underline{\theta}$. Absent adjustment costs, the long-run optima are

$$\underline{x} \in \arg\max_{x \in L} F(x, \underline{\theta}) \quad \text{and} \quad \bar{x} \in \arg\max_{x \in L} F(x, \bar{\theta}).$$

We assume that they satisfy $\underline{x} \leqslant \bar{x}$.[8] The agent initially chose $x_0 = \underline{x}$.

The agent is forward-looking, and discounts future payoffs by a factor of $\delta \in (0, 1)$. From the perspective of period $t = k$, given her period-$(k-1)$

[8]Given the properties of F, a standard result (Milgrom & Shannon, 1994, Theorem 4) guarantees the existence of such $\underline{x}$ and $\bar{x}$, provided the argmaxes are nonempty.

choice x_{k-1}, the agent's payoff from choosing $(x_t)_{t=k}^{\infty}$ going forward is

$$\mathcal{G}^k((x_t)_{t=k}^{\infty}, x_{k-1}, \bar{\theta}) = \mathcal{F}((x_t)_{t=k}^{\infty}, \bar{\theta}) - \mathcal{C}^k(x_{k-1}, (x_t)_{t=k}^{\infty}),$$

where

$$\mathcal{F}((x_t)_{t=k}^{\infty}, \bar{\theta}) = \sum_{t=k}^{\infty} \delta^{t-k} F(x_t, \bar{\theta})$$

and

$$\mathcal{C}^k(x_{k-1}, (x_t)_{t=k}^{\infty}) = \sum_{t=k}^{\infty} \delta^{t-k} C^t(x_t - x_{t-1}).$$

The agent's problem is to choose a sequence $(x_t)_{t=1}^{\infty}$ to maximize $\mathcal{G}^1(\cdot, x_0, \bar{\theta})$.

5.2 Dynamic le Chatelier principle

The following shows that our le Chatelier principle (Theorem 2) remains valid when the agent can adjust over time and is forward-looking: her 'short-run' choice x_t still satisfies $\underline{x} \leqslant x_t \leqslant \bar{x}$.

Proposition 3 (dynamic le Chatelier principle). *Suppose $F(x, \theta)$ is supermodular in x and has increasing differences in (x, θ), and that each C_i^t is single-dipped with $C_i(0) = 0$. Then provided the agent's problem admits a solution, there is a solution $(x_t)_{t=1}^{\infty}$ satisfying $\underline{x} \leqslant x_t \leqslant \bar{x}$ in every period t.*

When adjustment costs are time-invariant, a stronger dynamic le Chatelier principle holds: for any periods $t < T$ we have $\underline{x} \leqslant x_t \leqslant x_T \leqslant \bar{x}$, which is to say that the agent adjusts less in the short run than in the medium run. We shall use 'BCS' as shorthand for 'bounded on compact sets.'

Theorem 3 (strong dynamic le Chatelier principle). *Suppose $F(x,\theta)$ is supermodular and BCS in x and has increasing differences in (x,θ), and that $C_i^t = C_i$ for every period t and dimension i, where C_i is single-dipped and BCS with $C_i(0) = 0$. Then provided the agent's problem admits a solution, there is a solution $(x_t)_{t=1}^{\infty}$ satisfying $\underline{x} \leqslant x_t \leqslant x_T \leqslant \bar{x}$ for all periods $t < T$.*

The BCS requirements are mild, but they do imply that every adjustment has finite cost. Obviously continuity is a sufficient condition for BCS.

These results differ from our first two principal results, Theorems 1 and 2, in requiring cardinal rather than ordinal complementarity assumptions. Cardinality is needed for the usual reason, discussed at the end of section 4.

5.3 Application to pricing

An active literature in macroeconomics (e.g. Golosov & Lucas, 2007) examines the price stickiness central to the new Keynesian paradigm by studying fully forward-looking dynamic models of pricing subject to adjustment costs (usually called 'menu costs' in this context—see section 4.2). The basic mechanism is that non-convexities in adjustment costs give rise to price stickiness.

Theorem 3 delivers comparative statics for such pricing models, without any of the parametric assumptions that are typically placed on adjustment costs.[9] Consider again the monopoly pricing problem described in section 4.2, and recall that the monopolist's profit has increasing differences in (p,c), where p is her price and c is her marginal cost. Assume that the adjustment

[9]Common functional forms include quadratic (Rotemberg, 1982) and pure fixed cost (many papers, e.g. Caplin & Spulber, 1987; Caplin & Leahy, 1991; Golosov & Lucas, 2007).

cost C is single-dipped and BCS with $C(0) = 0$. Theorem 3 asserts that supply shocks cause inflation at every horizon: a one-off permanent increase of marginal cost leads prices to increase monotonically over time. The theorem furthermore provides that the path of prices does not overshoot the new frictionless monopoly price.

Although we phrased this finding in terms of a monopolist's pricing problem, it applies equally to the typical new Keynesian setting of monopolistic competition between many firms selling differentiated goods (see e.g. Galí, 2015). In that case, the demand curve in our analysis above is to be understood as *residual* demand, taking into account the other firms' pricing.

6 Application to capital investment

In the neoclassical theory of investment (originating with Jorgenson, 1963), a firm adjusts its capital stock optimally over time subject to adjustment costs. In the simplest such model, the profit of a firm with capital stock k_t is $F(k_t, (p, \eta, -r)) = pf(k_t, \eta) - rk_t$, where (p, r) are the prices of output and capital and $f(\cdot, \eta)$ is an increasing production function. Capital is subject to an adjustment cost: investing $i_t = k_t - k_{t-1}$ costs $C(i_t) \geqslant 0$, where $C(0) = 0$.

We assume that f has increasing differences, so that the parameter η shifts the marginal product of capital. Then $F(k, \theta)$ has increasing differences in (k, θ), where $\theta = (p, \eta, -r)$. Our discussion below extends directly to richer variants of this model featuring e.g. depreciation and time-varying prices.

The early literature assumed convex adjustment costs $C(\cdot)$, which yields gradual capital accumulation and an equivalence of the neoclassical theory with Tobin's (1969) 'q' theory of investment (see Hayashi, 1982). Later work focused on the 'lumpy' investment behavior that arises when adjustment costs are non-convex. 'Lumpiness' is empirically well-documented (see Cooper & Haltiwanger, 2006), and has implications for, among other things, business cycles (e.g. Thomas, 2002; Bachmann, Caballero, & Engel, 2013; Winberry, 2021) and the effects of microfinance programs on entrepreneurship in poor countries (e.g. Field, Pande, Papp, & Rigol, 2013; Bari, Malik, Meki, & Quinn, 2021).

Our comparative-statics theory handles both the convex case and rich forms of non-convexity. Our le Chatelier principles (the 'classical,' reduced-form Theorem 2 and the dynamic, forward-looking Theorem 3) are applicable provided merely that adjustment costs are single-dipped. Investment then increases at every horizon, and by more at longer horizons, whenever the marginal profitability of capital increases, whether due to a drop in its price r, a rise in the price p of output, or an increase of the marginal product of capital (an increase of η).

The aforementioned papers on microfinance consider models in which adjustment costs fail even to be single-dipped: there is a minimum investment size $I > 0$, meaning that investing $i \in (0, I)$ costs $C(i) = \infty$, (whereas investing $i \geqslant I$ has finite cost). Our fundamental result, Theorem 1, can accommodate such failures of single-dippedness: even with exotic adjustment costs like these, a rise in the marginal profitability of capital increases investment, just as would be the case if adjustment were costless.

All of these results generalize to multiple factors of production, on the pattern of section 4.1. It suffices to assume that the factors $x = (x_1, \ldots, x_n)$ are complements in production, meaning that the production function $f(x, \eta)$ is supermodular in x. Then, denoting factor prices by $r = (r_1, \ldots, r_n)$, the profit function $F(x, (p, \eta, -r)) = pf(x, \eta) - r \cdot x$ is supermodular in x, and has increasing differences in $(x, (p, \eta, -r))$ as before, so that all of our general results remain applicable. In case there are just $n = 2$ factors of production, the complementarity hypothesis may be replaced with substitutability (submodularity of $f(x, \eta)$ in x), using the trick described in section 4.1.

Appendix

A Standard definitions

A function $\psi : \mathbb{R} \to [0, \infty]$ is *single-dipped* exactly if there is an $x \in \mathbb{R}$ such that ψ is decreasing on $(-\infty, x]$ and increasing on $[x, \infty)$.

For $x, y \in \mathbb{R}^n$, we write $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$ and $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$. A set $L \subseteq \mathbb{R}^n$ is a *sublattice* of $\mathbb{R}^n$ exactly if for any $x, y \in L$, the vectors $x \wedge y$ and $x \vee y$ also belong to L.

Similarly, for any nonempty set $X \subseteq \mathbb{R}^n$, we write

$$\bigwedge X = \left(\inf_{x \in X} x_1, \ldots, \inf_{x \in X} x_n\right) \quad \text{and} \quad \bigvee X = \left(\sup_{x \in X} x_1, \ldots, \sup_{x \in X} x_n\right).$$

A set $L \subseteq \mathbb{R}^n$ is a *subcomplete sublattice* of $\mathbb{R}^n$ if for every nonempty $X \subseteq L$, the vectors $\bigwedge X$ and $\bigvee X$ belong to L. If a nonempty set $X \subseteq \mathbb{R}^n$ contains $\bigvee X$, we call $\bigvee X$ the *greatest element* of X, and denote it by

$\max X$. Similarly for the *least element,* denoted $\min X$.

Fix a sublattice L of $\mathbb{R}^n$. A function $\phi : L \to \mathbb{R}$ is called *supermodular* if $\phi(x) - \phi(x \wedge y) \leqslant \phi(x \vee y) - \phi(y)$ for any $x, y \in L$, *quasi-supermodular* if $\phi(x) - \phi(x \wedge y) \geqslant (>)\ 0$ implies $\phi(x \vee y) - \phi(y) \geqslant (>)\ 0$, and *(quasi-)submodular* if $-\phi$ is (quasi-)supermodular. Clearly supermodularity implies quasi-supermodularity. If $n = 1$, then every function $\phi : L \to \mathbb{R}$ is trivially supermodular.

Let Θ be a partially ordered set. A function $F : L \times \Theta \to \mathbb{R}$ has *(strict) increasing differences* if $F(y, \theta) - F(x, \theta)$ is (strictly) increasing in θ whenever $x \leqslant y$, has *single-crossing differences* if $F(y, \theta') - F(x, \theta') \geqslant (>)\ 0$ implies $F(y, \theta'') - F(x, \theta'') \geqslant (>)\ 0$ whenever $x \leqslant y$ and $\theta' \leqslant \theta''$, has *strict single-crossing differences* if $F(y, \theta') - F(x, \theta') \geqslant 0$ implies $F(y, \theta'') - F(x, \theta'') > 0$ whenever $x < y$ and $\theta' < \theta''$, and has *(strict) decreasing differences* if $-F$ has (strict) increasing differences. A function $F : L \times \Theta \to \mathbb{R}_{++}$ has *(strict) log increasing differences* exactly if $\ln F$ has (strict) increasing differences. (Strict) increasing differences and (strict) log increasing differences each imply (strict) single-crossing differences.

B Proof of Proposition 1

Let $\hat{x} \in \arg\max_{x \in L} G(x, \bar{\theta})$, and suppose toward a contradiction that $\hat{x} \ngeqslant \underline{x}$. Note that $\underline{x} \vee \hat{x} > \hat{x}$ and $\underline{x} > \underline{x} \wedge \hat{x}$. The proof of Theorem 1 yields $F(\underline{x} \vee \hat{x}, \bar{\theta}) \geqslant F(\hat{x}, \bar{\theta})$ and $C(\underline{x} \vee \hat{x} - \underline{x}) \leqslant C(\hat{x} - \underline{x})$, where the first inequality is strict if single-crossing differences is strict, and the second inequality is strict if reduction is costly. In either case, we have

$$F(\underline{x} \vee \hat{x}, \bar{\theta}) - C(\underline{x} \vee \hat{x} - \underline{x}) > F(\hat{x}, \bar{\theta}) - C(\hat{x} - \underline{x}),$$

which contradicts the fact that $\hat{x}$ maximizes $G(\cdot, \bar{\theta})$ on L. **QED**

C Proof of Proposition 2

The first inequality follows from Theorem 1, and the last one follows from Theorem 2. The middle inequality holds by Topkis's theorem (Topkis, 1978, Theorem 6.1) since $G(x, \bar{\theta}, h)$ is supermodular in x (as $F(x, \bar{\theta})$ is, and $C(\cdot, h)$ is additively separable) and has increasing differences in (x, h) (since each C_i has decreasing differences). **QED**

D Proof of Proposition 3

Let $(x_t)_{t=1}^{\infty}$ maximize $\mathcal{G}^1(\cdot, x_0, \bar{\theta})$. We shall show that $((x_t \vee \underline{x}) \wedge \bar{x})_{t=1}^{\infty}$ also maximizes $\mathcal{G}^1(\cdot, x_0, \bar{\theta})$; this suffices since $\underline{x} \leqslant (x_t \vee \underline{x}) \wedge \bar{x} \leqslant \bar{x}$ for each t.

We have $\mathcal{F}((\underline{x})_{t=1}^{\infty}, \underline{\theta}) \geqslant \mathcal{F}((x_t \wedge \underline{x})_{t=1}^{\infty}, \underline{\theta})$ since $\underline{x}$ maximizes $F(\cdot, \underline{\theta})$, which by supermodularity implies that $\mathcal{F}((x_t \vee \underline{x})_{t=1}^{\infty}, \underline{\theta}) \geqslant \mathcal{F}((x_t)_{t=1}^{\infty}, \underline{\theta})$, whence $\mathcal{F}((x_t \vee \underline{x})_{t=1}^{\infty}, \bar{\theta}) \geqslant \mathcal{F}((x_t)_{t=1}^{\infty}, \bar{\theta})$ by increasing differences. We also have $\mathcal{C}^1(x_0, (x_t \vee \underline{x})_{t=1}^{\infty}) \leqslant \mathcal{C}^1(x_0, (x_t)_{t=1}^{\infty})$ since for each dimension i and each period

t, C_i^t is single-dipped and minimized at 0, and

$$\begin{aligned} \text{either} \quad & x_{t,i} - x_{t-1,i} \leqslant \max\{x_{t,i}, \underline{x}_i\} - \max\{x_{t-1,i}, \underline{x}_i\} \leqslant 0 \\ \text{or} \quad & x_{t,i} - x_{t-1,i} \geqslant \max\{x_{t,i}, \underline{x}_i\} - \max\{x_{t-1,i}, \underline{x}_i\} \geqslant 0. \end{aligned}$$

Thus $(x_t^*)_{t=1}^\infty = (x_t \vee \underline{x})_{t=1}^\infty$ also maximizes $\mathcal{G}^1(\cdot, x_0, \bar{\theta})$.

Since $\bar{x}$ maximizes $F(\cdot, \bar{\theta})$, we have $\mathcal{F}((\bar{x})_{t=1}^\infty, \bar{\theta}) \geqslant \mathcal{F}((x_t^* \vee \bar{x})_{t=1}^\infty, \bar{\theta})$, so that $\mathcal{F}((x_t^* \wedge \bar{x})_{t=1}^\infty, \bar{\theta}) \geqslant \mathcal{F}((x_t^*)_{t=1}^\infty, \bar{\theta})$ by supermodularity. As above, we have $\mathcal{C}^1(x_0, (x_t^* \wedge \bar{x})_{t=1}^\infty) \leqslant \mathcal{C}^1(x_0, (x_t^*)_{t=1}^\infty)$. Thus $(x_t^* \wedge \bar{x})_{t=1}^\infty = ((x_t \vee \underline{x}) \wedge \bar{x})_{t=1}^\infty$ also maximizes $\mathcal{G}^1(\cdot, x_0, \bar{\theta})$, as claimed. **QED**

E Proof of Theorem 3

Note that under the theorem's hypotheses, the functions $\mathcal{C}^t$ and $\mathcal{G}^t$ are the same in each period t, so may be written simply as $\mathcal{C}$ and $\mathcal{G}$.

For any sequence $\boldsymbol{x} = (x_t)_{t=1}^\infty$ in L and any $k \in \mathbb{N}$, let $S_k\boldsymbol{x}$ denote the sequence in L whose t^{th} entry is x_t for $t < k$ and $x_{t-1} \vee x_t$ for $t \geqslant k$.

Assume that the agent's problem admits a solution. Let $\boldsymbol{x^1} = (x_t^1)_{t=1}^\infty$ be a solution satisfying $\underline{x} \leqslant x_t^1 \leqslant \bar{x}$ in every period t; such a solution exists by Proposition 3. Define $X_t = x_1^1 \vee x_2^1 \vee \cdots \vee x_{t-1}^1 \vee x_t^1$ for $t \in \mathbb{N}$, and $X_0 = \underline{x}$.

Write $\boldsymbol{x^k} = S_k S_{k-1} \cdots S_3 S_2 \boldsymbol{x}$ for $k \geqslant 2$. By inspection, the first k entries of $\boldsymbol{x^k}$ are $X_1, X_2, \ldots, X_{k-1}, X_k$. Clearly $\underline{x} \leqslant X_t \leqslant X_T \leqslant \bar{x}$ for any periods $t < T$. To prove the theorem, we need only show that $\boldsymbol{x}^\infty = (X_1, X_2, X_3, \ldots)$ is optimal.

It suffices to show for each $k \in \mathbb{N}$ that $\boldsymbol{x^k}$ is optimal. For then, letting V be the optimal value and noting that both $\boldsymbol{x^k} = (x_t)_{t=1}^{\infty}$ and $\boldsymbol{x}^{\infty}$ have $X_1, \ldots, X_k$ as their first k entries, we have

$$\begin{aligned}\left|\mathcal{G}(\boldsymbol{x}^{\infty}, x_0, \bar{\theta}) - V\right| &= \left|\mathcal{G}(\boldsymbol{x}^{\infty}, x_0, \bar{\theta}) - \mathcal{G}(\boldsymbol{x^k}, x_0, \bar{\theta})\right| \\ &= \delta^k \left|\mathcal{G}((X_t)_{t=k+1}^{\infty}, X_k, \bar{\theta}) - \mathcal{G}((x_t)_{t=k+1}^{\infty}, X_k, \bar{\theta})\right|.\end{aligned}$$

By BCS, the right-hand '$|\cdot|$' is bounded uniformly over $k \in \mathbb{N}$,[10] and so the right-hand side vanishes as $k \to \infty$, yielding $\mathcal{G}(\boldsymbol{x}^{\infty}, x_0, \bar{\theta}) = V$.

To show that $\boldsymbol{x^k}$ is optimal for each $k \in \mathbb{N}$, we employ induction on $k \in \mathbb{N}$. The base case $k = 1$ is immediate.

For the induction step, fix any $k \in \mathbb{N}$, and suppose that $\boldsymbol{x^k} = (x_t)_{t=1}^{\infty}$ is optimal; we will show that $\boldsymbol{x^{k+1}} = S_{k+1}\boldsymbol{x^k}$ is also optimal. Let $(\widetilde{x}_t)_{t=1}^{\infty}$ be the sequence with t^{th} entry x_t for $t < k$ and $x_t \wedge x_{t+1}$ for $t \geqslant k$. Since $\boldsymbol{x^k} = (x_t)_{t=1}^{\infty}$ is optimal, and $(\widetilde{x}_t)_{t=1}^{\infty}$ shares its first $k-1$ entries $X_1, \ldots, X_{k-1}$, we have

$$\mathcal{G}((x_t)_{t=k}^{\infty}, X_{k-1}, \bar{\theta}) \geqslant \mathcal{G}((\tilde{x}_t)_{t=k}^{\infty}, X_{k-1}, \bar{\theta}),$$

which may be written in full as

$$\begin{aligned}&\sum_{t=k}^{\infty} \delta^{t-k} \left[F(x_t, \bar{\theta}) - F(x_t \wedge x_{t+1}, \bar{\theta})\right] \\ &\qquad - \sum_{t=k}^{\infty} \delta^{t-k} \left[C(x_t - x_{t-1}) - C(x_t \wedge x_{t+1} - x_{t-1} \wedge x_t)\right] \geqslant 0. \qquad (1)\end{aligned}$$

[10] Since X_t and x_t belong to the compact set $[\underline{x}, \bar{x}]$ in every period t, there are constants $A, B > 0$ such that $|F(X_t, \bar{\theta}) - F(x_t, \bar{\theta})| \leqslant 2A$ and $|C(X_t - X_{t-1}) - C(x_t - x_{t-1})| \leqslant 2B$ for all t, so the right-hand '$|\cdot|$' is bounded by $2(A+B)/(1-\delta)$.

(Note that since $x_t = X_t$ for every $t \leqslant k$, we have $x_{k-1} \wedge x_k = X_{k-1} = x_{k-1}$.) Since $F(\cdot, \bar{\theta})$ is supermodular, it holds for every $t \geqslant k$ that

$$F(x_t \vee x_{t+1}, \bar{\theta}) - F(x_{t+1}, \bar{\theta}) \geqslant F(x_t, \bar{\theta}) - F(x_t \wedge x_{t+1}, \bar{\theta}) \tag{2}$$

We furthermore claim that for each $t \geqslant k$,

$$\begin{aligned} C(x_t \vee x_{t+1} - x_{t-1} \vee x_t) - C(x_{t+1} - x_t) \\ \leqslant C(x_t - x_{t-1}) - C(x_t \wedge x_{t+1} - x_{t-1} \wedge x_t); \end{aligned} \tag{3}$$

we shall prove this shortly. Combining (1), (2) and (3), and changing variables in the sums, we obtain

$$\begin{aligned} \sum_{t=k+1}^{\infty} \delta^{t-(k+1)} \left[F(x_{t-1} \vee x_t, \bar{\theta}) - F(x_t, \bar{\theta})\right] \\ - \sum_{t=k+1}^{\infty} \delta^{t-(k+1)} \left[C(x_{t-1} \vee x_t - x_{t-2} \vee x_{t-1}) - C(x_t - x_{t-1})\right] \geqslant 0. \end{aligned}$$

By inspection, this says precisely that $(\hat{x}_t)_{t=1}^{\infty} = \boldsymbol{x^{k+1}} = S_{k+1}\boldsymbol{x^k}$ satisfies

$$\mathcal{G}((\hat{x}_t)_{t=k+1}^{\infty}, X_k, \bar{\theta}) \geqslant \mathcal{G}((x_t)_{t=k+1}^{\infty}, X_k, \bar{\theta}).$$

(Note that since $x_t = X_t$ for every $t \leqslant k$, we have $x_{(k+1)-2} \vee x_{(k+1)-1} = X_{k-1} \vee X_k = X_k = x_{(k+1)-1}$.) Since $\boldsymbol{x^{k+1}} = (\hat{x}_t)_{t=1}^{\infty}$ and $\boldsymbol{x^k} = (x_t)_{t=1}^{\infty}$ agree in their first k entries, and $\boldsymbol{x^k}$ is optimal, it follows that $\boldsymbol{x^{k+1}}$ is optimal, too.

It remains to show that (3) holds. It suffices to prove for each i that

$$C_i(b \vee c - a \vee b) + C_i(b \wedge c - a \wedge b) \leqslant C_i(b-a) + C_i(c-b) \quad \text{for any } a, b, c. \tag{4}$$

(We've renamed $x_{t-1,i} = a$, $x_{t,i} = b$ and $x_{t+1,i} = c$.) When b is not extreme (neither least nor greatest), (4) holds trivially because the left-hand side is equal to the right-hand side. When b is extreme, (4) reads

$$C_i(c-a) \leqslant C_i(b-a) + C_i(c-b),$$

and we have

$$\begin{array}{rrlcrl} \text{either} & \text{(i)} & 0 \leqslant c-a \leqslant b-a & \text{or} & \text{(ii)} & 0 \leqslant c-a \leqslant c-b \\ \text{or} & \text{(iii)} & 0 \geqslant c-a \geqslant b-a & \text{or} & \text{(iv)} & 0 \geqslant c-a \geqslant c-b. \end{array}$$

Since C_i is single-dipped, minimized at zero and non-negative, we have $C_i(c-a) \leqslant C(b-a) \leqslant C(b-a) + C(c-b)$ in the first and third cases, and $C_i(c-a) \leqslant C(c-b) \leqslant C(b-a) + C(c-b)$ in the second and fourth. **QED**

Condition (4) is a special case (with $b = d$) of the following condition:

$$C_i(b \vee c - a \vee d) + C_i(b \wedge c - a \wedge d) \leqslant C_i(b-a) + C_i(c-d) \quad \text{for any } a, b, c, d.$$

This says that the map $(a, b) \mapsto C_i(a-b)$ is submodular, which holds if and only if C_i is convex. Remarkably, we did not have to assume convexity to obtain a monotone adjustment path—single-dippedness was enough.

References

Antić, N., & Salant, Y. (in progress). *Monopoly theory with price-increase aversion.*

Bachmann, R., Caballero, R. J., & Engel, E. M. R. A. (2013). Aggregate implications of lumpy investment: New evidence and a DSGE model. *American Economic Journal: Macroeconomics*, *5*(4), 29–67. https://doi.org/10.1257/mac.5.4.29

Bari, F., Malik, K., Meki, M., & Quinn, S. (2021). *Asset-based microfinance for microenterprises: Evidence from Pakistan* [working paper, Jan 2021]. https://doi.org/10.2139/ssrn.3778754

Berger, D., & Vavra, J. (2015). Consumption dynamics during recessions. *Econometrica*, *83*(1), 101–154. https://doi.org/10.3982/ECTA11254

Caplin, A., & Leahy, J. (1991). State-dependent pricing and the dynamics of money and output. *Quarterly Journal of Economics*, *106*(3), 683–708. https://doi.org/10.2307/2937923

Caplin, A., & Spulber, D. F. (1987). Menu costs and the neutrality of money. *Quarterly Journal of Economics*, *102*(4), 703–725. https://doi.org/10.2307/1884277

Cooper, R. W., & Haltiwanger, J. C. (2006). On the nature of capital adjustment costs. *Review of Economic Studies*, *73*(3), 611–633. https://doi.org/10.1111/j.1467-937X.2006.00389.x

Field, E., Pande, R., Papp, J., & Rigol, N. (2013). Does the classic microfinance model discourage entrepreneurship among the poor? Experimental evidence from India. *Journal of Political Economy*, *103*(6), 2196–2226. https://doi.org/10.1257/aer.103.6.2196

Galí, J. (2015). *Monetary policy, inflation, and the business cycle: An introduction to the new Keynesian framework and its applications* (2nd). Princeton University Press.

Golosov, M., & Lucas, R. E., Jr. (2007). Menu costs and Phillips curves. *Journal of Political Economy*, *115*(2), 171–199. https://doi.org/10.1086/512625

Grossman, S. J., & Laroque, G. (1990). Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. *Econometrica*, *58*(1), 25–51. https://doi.org/10.2307/2938333

Hayashi, F. (1982). Tobin's marginal q and average q: A neoclassical interpretation. *Econometrica*, *50*(1), 213–224. https://doi.org/10.2307/1912538

Jorgenson, D. W. (1963). Capital theory and investment behavior. *American Economic Review*, *53*(2), 247–259.

Kaplan, G., & Violante, G. L. (2014). A model of the consumption response to fiscal stimulus payments. *Econometrica*, *82*(4), 1199–1239. https://doi.org/10.3982/ECTA10528

Levy, D., Bergen, M., Dutta, S., & Venable, R. (1997). The magnitude of menu costs: Direct evidence from large U.S. supermarket chains. *Quarterly Journal of Economics*, *112*(3), 791–824. https://doi.org/10.1162/003355397555352

Mankiw, N. G. (1985). Small menu costs and large business cycles: A macroeconomic model of monopoly. *Quarterly Journal of Economics*, *100*(2), 529–538. https://doi.org/10.2307/1885395

Milgrom, P., & Roberts, J. (1990). Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, *58*(6), 1255–1277. https://doi.org/10.2307/2938316

Milgrom, P., & Roberts, J. (1996). The LeChatelier principle. *American Economic Review*, *86*(1), 173–179.

Milgrom, P., & Shannon, C. (1994). Monotone comparative statics. *Econometrica*, *62*(1), 157–180. https://doi.org/10.2307/2951479

Rotemberg, J. J. (1982). Monopolistic price adjustment and aggregate output. *Review of Economic Studies*, *49*(4), 517–531. https://doi.org/10.2307/2297284

Samuelson, P. A. (1947). *Foundations of economic analysis*. Harvard University Press.

Thomas, J. K. (2002). Is lumpy investment relevant for the business cycle? *Journal of Political Economy*, *110*(3), 508–534. https://doi.org/10.1086/339746

Tobin, J. (1969). A general equilibrium approach to monetary theory. *Journal of Money, Credit and Banking*, *1*(1), 15–29. https://doi.org/10.2307/1991374

Topkis, D. M. (1978). Minimizing a submodular function on a lattice. *Operations Research*, *26*(2), 305–321. https://doi.org/10.1287/opre.26.2.305

Topkis, D. M. (1998). *Supermodularity and complementarity.* Princeton University Press.

Winberry, T. (2021). Lumpy investment, business cycles, and stimulus policy. *American Economic Review*, *111*(1), 364–396. https://doi.org/10.1257/aer.20161723